Some Remarks on Rough Inclusions

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Abstract. Rough inclusions are primitive notions of Rough Mereology, obtained by 'fuzzification' in a sense of the part relation which is the primitive notion of Mereology. Mereology goes back as an ideology to Aristotle's *Ta Meta Ta Fysica*, Book Δ, 1203b where the notion of a part was first outlined; a formal rendering of parts was achieved by S. Leśniewski (1916) who built a formal edifice of Mereology. Especially suited to reasoning about mass categories, e.g., concepts, figures, solids, etc., Mereology has become one of main tools for Spatial Reasoning, cf., Tarski's axiomatization of geometry of solids (1929), RCC calculi, Mereotopology in analysis of boundaries, etc. Rough Mereology goes a step further by considering the relation of a part to a degree, hence, with a liberty of language, one may set that *Rough Mereology to Mereology = Fuzzy Set Theory to Na ive Set Theory.* Implementations of the idea of a part to a degree are *rough inclusions.* Their study is one of principal problems in Rough Mereology. In this contribution, we establish topological conditions which guarantee that rough inclusions which satisfy them are of a particular form, e. g., they are residual implications induced by the Łukasiewicz t–norm. In what follows, basic notions are introduced and basic auxiliary results recalled, culminating in the aforementioned result.

1 Rough inclusions

A *rough inclusion* is a ternary relation $\mu(x, y, r)$ on a set U of objects. A preliminary to existence of the rough inclusion is that U be a mereological universe, i.e., a part relation π is introduced on U with the associated ingredient relation $ingr$ [3]; to recall, the relation π is transitive and irreflexive, and the relation $ingr$ is defined as the union $\pi \cup eq$ where eq is the equality relation. A technical means for reasoning about parts is the Leśniewski inference rule: *given x, y, if for each t with* $ingr(t, x)$ *there exists z such that* $ingr(z, t)$ *and* $ingr(z, y)$*, then* $ingr(x, y)$.

Now, the relation $\mu(x, y, r)$, under the generic name of *rough inclusion*, cf., [6], is introduced as satisfying on the set U the following conditions

RINC1 $\mu(x, y, 1) \Leftrightarrow ingr(x, y)$

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RINC2 $\mu(x, y, 1) \Rightarrow \forall z[\mu(z, x, r) \Rightarrow \mu(z, y, r)]$

RINC3 $\mu(x, y, r) \wedge s < r \Rightarrow \mu(x, y, s)$

The immediate consequences of postulates RINC1–RINC3 are

RINC4 $\mu(x, x, 1)$

RINC5 $\mu(x, y, 1) \wedge \mu(y, z, 1) \Rightarrow \mu(x, z, 1)$

RINC6 $\mu(x, y, 1) \wedge \mu(y, x, 1) \Leftrightarrow x = y$

RINC7 $x \neq y \Rightarrow \neg\mu(x, y, 1) \vee \neg\mu(y, x, 1)$

RINC8 $\forall z \forall r[\mu(z, x, r) \Leftrightarrow \mu(z, y, r)] \Rightarrow x = y$

By a *model* for rough mereology, we mean a quadruple

$$M = (V_M, \pi_M, ingr_M, \mu_M)$$

where V_M is a set with a part relation $\pi_M \subseteq V_M \times V_M$, the associated ingredient relation $ingr_M = \pi_M \cup eq \subseteq V_M \times V_M$, and a rough inclusion $\mu_M \subseteq V_M \times V_M \times [0, 1]$ which satisfies RINC1–RINC3 with respect to $ingr_M$.

2 T–norms and residual implications vs. rough inclusions

We recall that a t–norm T is a function $T : [0, 1]^2 \to [0, 1]$, which satisfies the following postulates

TN1 T is *associative*: $T(T(x, y), z) = T(x, T(y, z))$

TN2 T is *commutative*: $T(x, y) = T(y, x)$

TN3 T is *non–decreasing in each coordinate*: $T(z, y) \geq T(x, y)$ *whenever* $z \geq x$

TN4 $T(1, x) = x$

TN5 $T(x, 0) = 0$

Moreover, T may satisfy an additional postulate

TN6 T *is continuous*

'Classical' t–norms

the Łukasiewicz t–norm $L(x,y) = max\{0, x+y-1\}$,

the Product t–norm $P(x,y) = x \cdot y$,

the Minimum t–norm $M(x,y) = min\{x,y\}$,

satisfy TN1–TN6. Let us observe that L and P do satisfy

TN7 $T(x,x) < x$ *for each* $x \in (0,1)$

A *t*–norm $T(x,y)$ which satisfies postulates TN6, TN7 is said to be an *Archimedean t–norm.*

Given a *t*–norm T, the *residual implication* $\Rightarrow_T$ is defined via the condition

$$RI\ x \Rightarrow_T y \geq z \Leftrightarrow T(x,z) \leq y \tag{1}$$

The following properties of residual implications follow.

Proposition 1. *For every t–norm T, the residual implication $\Rightarrow_T$ satisfies the following conditions*

RI1 $T(x,y) \leq z \Rightarrow_T u$ *if and only if* $x \leq T(y,z) \Rightarrow_T u$

RI2 $y \Rightarrow_T (z \Rightarrow_T u) = T(y,z) \Rightarrow_T u$

RI3 $y \Rightarrow_T z = 1$ *if and only if* $y \leq z$

RI4 $x \leq y \Rightarrow_T u$ *if and only if* $y \leq x \Rightarrow_T u$

RI5 $y \leq (y \Rightarrow_T z) \Rightarrow_T z$

RI6 $T(x,y) \Rightarrow_T 0 = x \Rightarrow_T (y \Rightarrow_T 0)$

RI7 $x \Rightarrow_T (y \Rightarrow_T r) \geq z$ *if and only if* $(z \Rightarrow_T x) \Rightarrow_T r \geq y$

RI8 $x \Rightarrow_T 0 \geq 0$

Proof. Properties RI1–RI4 follow straightforwardly by definition. We comment a bit on the last four. As $y \Rightarrow_T z \leq y \Rightarrow_T z$ we have $T(y \Rightarrow_T z, y) \leq z$, hence, by commutativity of T it follows that $T(y, y \Rightarrow_T z) \leq z$ which implies $y \leq (y \Rightarrow_T z) \Rightarrow_T z$, i.e., RI5 follows.

Associativity of T implies that $z \leq T(x,y) \Rightarrow_T 0$ if and only if $T(z, T(x,y)) \leq 0$, i.e., $T(T(z,x),y) \leq 0$ so equivalently $T(z,x) \leq y \Rightarrow_T 0$ which is equivalent to $z \leq x \Rightarrow_T (y \Rightarrow_T 0)$. From the equivalence of the first and the last statements RI6 follows. Property RI7 is a paraphrase in terms of $\Rightarrow_T$ of associativity of T whereas RI8 does express the property that $T(x,0) = 0$ □

Residual implications induce rough inclusions by means of the following

Proposition 2. *Given a residual implication* $\Rightarrow_T$*, the equivalence* $\mu_T(x,y,r)$ *if and only if* $x \Rightarrow_T y \geq r$ *defines a rough inclusion* μ_T*.*

In particular, basic rough inclusions induced by residual implications of t–norms L, P, M, respectively, are

$\mu_L(x,y,r)$ if and only if $min\{1, 1-x+y\} \geq r$,

$\mu_P(x,y,r)$ if and only if $x > y$ and $\frac{y}{x} \geq r$ *else* $r = 1$,

$\mu_M(x,y,r)$ if and only if $x > y$ and $y \geq r$ *else* $r = 1$.

3 Necessary topology

It follows by Proposition 2, that each rough inclusion $\mu(x,y,r)$ can be regarded as a set–valued mapping, its values r in a convex subset of the interval [0, 1] containing 0; the topology in question is therefore the topology of set–valued mappings. For those mappings a notion of continuity is fused from two notions of semi–continuity; a mapping $f : X \rightarrow 2^Y$ is upper–semi–continuous (usc) (respectively, lower–semi–continuous (lsc)) if and only if for each open set $P \subseteq Y$, the set $f^{\leftarrow}(P) = \{x : f(x) \subseteq P\}$ is open in X (respectively, the set $f^{\cap}(P) = \{x : f(x) \cap P \neq \emptyset\}$ is open in X). The mapping f is continuous if and only if it is upper– and lower–semi–continuous.

As, clearly, every mapping $f : X \rightarrow Y$ is a set–valued mapping with singleton values, the above notions can be easily translated into semi–continuity notions for those ordinary mappings: the mapping $f : X \rightarrow Y$ is upper–semi–continuous (respectively, lower–semi–continuous) if and only if for every r, the set $\{x : f(x) \geq r\}$ is closed (respectively, for every r, the set $\{x : f(x) \leq r\}$ is closed).

4 Mostert–Shield, Faucett, and Menu–Pavelka theorems

From theorems of Mostert and Shields [5], respectively, Faucett [1] on topological semigroups, important consequences for t–norms follow, cf., also Hajék [2]. These results depend on the notion of a nilpotent element; an element $x \in (0,1]$ is *nilpotent* for a t–norm T if and only if $T^n(x) = 0$ for some n, where $T^{n+1}(x) = T(x, T^n(x))$ and $T^1(x) = T(x,x)$; one can easily see that $L^n(x) = 0$ if and only if $x \leq \frac{n}{n+1}$ hence each $x \in (0,1)$ is nilpotent whereas $P^n(x) = 0$ with some n if and only if $x = 0$. An element $x \in [0,1]$ is an *idempotent* for a t–norm T if and only if $T(x,x) = x$. Of the three classical t–norms introduced by us, only the minimum $M(x,y)$ has idempotents distinct from $0, 1$.

By the theorem due to Mostert and Shields, each continuous t–norm T free of idempotents other than $0, 1$ with at least one nilpotent element is isomorphic to the t–norm L, and by the Faucett theorem each continuous t–norm T free of idempotents other than $0, 1$ without any nilpotent element is isomorphic to the t–norm P, where an isomorphism between t–norms T and T' means that for some automorphism $f : [0, 1] \to [0, 1]$, one has $T'(x, y) = f^{-1}(T(f(x), f(y))$.

We may now characterize in topological terms t–norms and residual implications by a result due to Menu and Pavelka [4].

Proposition 3. *(Menu–Pavelka) Any t – norm T is lower–semi–continuous; moreover, any associative and commutative function T with $T(0, x) = 0, T(1, x) = x$ which is lower–semi–continuous is a t–norm. In this case the residual implication $\Rightarrow_T$ is given by the condition $y \Rightarrow_T z = max\{x : T(x, y) \leq z\}$.*

Proof. We comment for completeness' sake on this statement. Let us fix a t–norm T, along with a real number $r \in [0, 1]$ and consider the set

$$M = \{(x, y) : T(x, y) \leq r\} \tag{2}$$

We want to show that M is closed. First, we recall the observation in Menu–Pavelka, op. cit., that the restricted function $T(x, .)$ of one variable does preserve suprema for each x. Consider a sequence $(z_n)_n$ increasing to a limit z, let $T(x, z_n) = y_n$, and $T(x, z) = y$; let as well $y^0 = sup_n y_n$, hence, as $y \geq y_n$, each n, one has $y \geq y^0$. On the other hand, as $T(x, z_n) \leq y^0$, we have $x \Rightarrow y^0 \geq z_n$, each n, hence, $x \Rightarrow y^0 \geq z$, i.e., $T(x, z) \leq y^0$ implying that $y \leq y^0$, so finally $y = y^0$.

Assume, now, to the contrary, that $(x_0, y_0) \in [0, 1]^2$ and a sequence $(x_n, y_n)_n$ exists such that

(i) $T(x_0, y_0) > r$
(ii) $T(x_n, y_n) \leq r$ *for each* n
(iii) $(x_0, y_0) = lim_n(x_n, y_n)$

Then, by monotonicity of T, we have for each n that either $x_n < x$ or $y_n < y$; we may assume that the set $\{n : x_n < x_0\}$ is infinite and in consequence we may assume that $x_n < x_0$ for each n.

Letting $\overline{x}_n = max\{x_1, .., x_n\}$ and $\underline{y}_n = min\{y_1, ..., y_n\}$ we have $T(\overline{x}_n, \underline{y}_n) \leq r$, so we may assume that the sequence $(x_n)_n$ is increasing to x_0 and the sequence $(y_n)_n$ is decreasing to y_0.

As $sup_n T(x_n, y_n) = T(x_0, y_0)$ it follows by monotonicity of T, along with (ii), (iii), that $T(x_0, y_0) \leq r$, a contradiction. Thus M is closed and a fortiori T is lower–semi–continuous.

The converse may be proved along the same lines. As a corollary to the lower–semi–continuity of the t – norm T, we have that the set $M = \{(x, y) : T(x, y) \leq z\}$ is closed hence the set $\{x : x \leq y \Rightarrow_T z\}$ is closed hence it contains its greatest element and thus $y \Rightarrow_T z = max\{x : T(x, y) \leq z\}$ □

We denote with the symbol $\overline{\mu}(x, y)$ the maximal value of r such that $\mu(x, y, r)$ does hold, existing in virtue of Proposition 3.

The question which t–norms may have continuous residuals was settled in Menu and Pavelka [4].

Proposition 4. *(Menu–Pavelka, op. cit.) For every t–norm T, continuity of $\Rightarrow_T$ implies continuity of T. In this case, T is isomorphic to L.*

Proof. (After Menu–Pavelka, op. cit.) We assume that $\Rightarrow_T$ is continuous; we first verify the following

Claim. *Consider the function* $x \Rightarrow_T a$ on the interval $[0,1]$ *with* $0 \leq a < 1$. *Then*

$$y = (y \Rightarrow_T a) \Rightarrow_T a$$

for each $y \in [a, 1]$

To verify Claim, we recall that

$$(i)\ y \leq (y \Rightarrow_T a) \Rightarrow_T a$$

by RI5; it remains to show that in case $y \geq a$, we have

$$(ii)\ y \geq (y \Rightarrow_T a) \Rightarrow_T a$$

As the function $q(x) = x \Rightarrow_T a$ is continuous decreasing with $q(a) = 1, q(1) = a$, there is $z \geq a$ with $y = z \Rightarrow_T a$. Then

$$\begin{cases} y = z \Rightarrow_T a \geq (((z \Rightarrow_T a) \rightarrow_T a) \Rightarrow_T a) = \\ (y \Rightarrow_T a) \Rightarrow_T a \end{cases} \tag{3}$$

by the fact that the superposition $(x \Rightarrow_T a) \Rightarrow_T a$ is increasing so (ii) follows, and (i) and (ii) together yield Claim.

From Claim it follows with $a = 0$ that

$$(iii)\ T(x, y) = (T(x, y) \Rightarrow_T 0) \Rightarrow_T 0 = ((x \Rightarrow_T (y \Rightarrow_T 0) \Rightarrow_T 0$$

where the last equality holds by RI6. Hence, by (iii), T is definable in terms of superpositions of $\Rightarrow_T$ and thus T is continuous.
We now solve the equation

$$(iv)\ T(x, x) = x$$

Assume that (iv) holds with $x \neq 0, 1$. Take $c \leq x \leq d$, so there is u with $c = T(x, u)$. Then

$$T(x, c) = T(x, T(x, u)) = T(T(x, x), u) = T(x, u) = c$$

and we have

$$c = T(x, c) \leq T(d, c) \leq T(1, c) = c$$

i.e., $T(d, c) = c$, hence, $d \rightarrow_T c = c$ for each pair d, c with $c \leq x \leq d$.

This means that for $c < x$, the function $\Rightarrow_T c$ is not any injection, contrary to Claim. We infer that the equation $T(x,x) = x$ has only $0, 1$ as solutions.

To conclude the proof one has to refer to the mentioned above results of Mostert and Shields [5] and Faucett [1]. By these results, T is equivalent either to L or to P; however, P has to be excluded as its residual implication is discontinuous □

5 Some results on rough inclusions

We begin with a dual proposition, referring to residual implications, whose some elements can be traced in Menu–Pavelka [4].

Proposition 5. *Each function $\phi(x,y) : [0,1]^2 \to [0,1]$ which is non–increasing in the first argument x, non–decreasing in the second argument y, and it does satisfy conditions RI3, RI4, RI7, RI8, is of the form $\Rightarrow_T$ for some t–norm T, and it is necessarily upper–semi–continuous.*

Proof. The function $T(x,y)$ defined by the duality (i) $T(x,y) \leq r \Leftrightarrow \phi(x,r) \geq y$ satisfies conditions T1–T5, as observed earlier, RI4, RI7 do express commutativity and associativity of T, and RI3, RI8 are responsible for boundary conditions for T. Hence, duality (i) qualifies ϕ as the residuum $\Rightarrow_T$ which is upper–semi–continuous by the same duality and lower–semi–continuity of T demonstrated in Proposition 3 □

To state the next result, we need some notational conventions. We know that each rough inclusion μ_T takes as its values closed intervals of the form $[0, \overline{\mu}(x,y)]$. We use the symbol $\overline{\mu}(x,y) \ni r$ for the fact that $\mu(x,y,r)$.

Proposition 6. *Each function $\mu(x,y) : [0,1]^2 \to 2^{[0,1]}$, whose values are closed intervals $[0, \overline{\mu}(x,y)]$, which satisfies conditions*

MI1 $\overline{\mu}(x,y) \ni 1$ *if and only if* $x \leq y$

MI2 $\overline{\mu}(y,u) \ni x$ *if and only if* $\overline{\mu}(x,u) \ni y$

MI3 $\overline{\mu}(x, \overline{\mu}(y,r)) \ni z$ *if and only if* $\overline{\mu}(\overline{\mu}(z,x), r) \ni y$

MI4 $\overline{\mu}(x,0) \ni 0$

and is non–increasing in x and non–decreasing in y is of the form μ_T for some t–norm T.

Proof. We let $\phi(x,y) \geq r$ if and only if $\overline{\mu}(x,y) \ni r$; it is straightforward to verify that conditions MI1–MI4 correspond to conditions RI3, RI4, RI7, RI8 for ϕ and accordingly, ϕ is of the form $\Rightarrow_T$ by Proposition 5, hence, μ is μ_T □

Following this line, we consider μ_T as a multi–valued mapping; we prove its semi–continuity (upper) as of a multi–mapping.

Proposition 7. *The set* $E_T(s) = \{(x,y) \in [0,1]^2 : \overline{\mu}(x,y) < s\}$ *is open for each* s, *i.e.,* μ_T *is upper–semi–continuous as a many–valued mapping.*

Proof. It follows by upper–semi–continuity of μ as a single-valued map □

Proposition 4 has a counterpart for rough inclusions.

Proposition 8. *Each rough inclusion* μ_T *continuous as a multi–valued mapping is isomorphic to* μ_L *in the sense that* T *is isomorphic to* L.

Proof. We consider the corresponding residuum $x \Rightarrow_T y = \overline{\mu}(x,y)$ and prove its continuity. Let $\overline{\mu}(x,y) \in U$ for an open set U; let $\overline{\mu}(x,y) \in (a,b) \subseteq U$. Let $V = [0,a] \cup (a,b)$, and consider $W = \{(x,y) : [0,\overline{\mu}(x,y)] \subseteq V$ and $[0,\overline{\mu}(x,y)] \cap (a,b) \neq \emptyset\}$; clearly, W is open by continuity of μ_T and $(x,y) \in W$ if and only if $x \Rightarrow_T y \in U$, hence, $x \Rightarrow_T y$ is continuous and Proposition 4 implies that T is isomorphic to L □

References

1. Faucett, W.M.: Compact semigroups irreducibly connected between two idempotents. Proc. Amer. Math.Soc. 6, pp 741 – 747 (1955)
2. Hájek, P.: Metamathematics of Fuzzy Logic. Kluwer, Dordrecht (1998)
3. Leśniewski, S.: Srzednicki J., Surma S. J., Barnett D., Rickey V. F. (Eds.) Collected Works of Stanisław Leśniewski. Kluwer, Dordrecht (1992)
4. Menu, J., Pavelka, J.: A note on tensor products on the unit interval. Comm. Univ. Carolinae 17, pp 71 –83 (1976)
5. Mostert, P. S., Shields, A. L.: On the structure of semigroups on a compact manifold with boundary. Ann. Math. 65, pp 117 – 143 (1957)
6. Polkowski, L.: Approximate Reasoning by Parts. An Introduction to Rough Mereology. Springer Verlag, Berlin (2011)