Relating Categorical Semantics for Higher Dimensional Automata*

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Abstract The intention of the paper is to show how several categorical (open maps, path-bisimilarity and coalgebraic) approaches to an abstract characterization of bisimulation relate to each other and to hereditary history preserving bisimulation, in the setting of higher dimensional automata. Such a relating makes it possible to develop a metatheory designed for unified definition and study of equivalences in true concurrency semantics.

1 Introduction

Geometrical methods in concurrency theory have appeared recently for modelling, analysis and verification of the behaviour of concurrent systems. The most popular geometric model for concurrency is higher dimensional automata (HDA) which have been proposed by V. Pratt [19]. Actually at about the same time a bisimulation semantics has been given for HDA in [6]. Based on the concepts of HDA, numerous papers have emerged in the literature. Basic strands of research are concerned with giving true concurrent semantics to concurrent languages [10,8,3], with analyzing correctness of distributed databases [4], with formalizing the fault-tolerant implementation of distributed programs [11,9,12]. The relationships between higher dimensional automata and other true concurrent models have been thoroughly studied in the paper [7].

In order to unify and clarify apparent differences between the extensive amount of research within the field of behavioral equivalences, several category-theoretic approaches to the matter have appeared. Two of them were initiated by Joyal, Nielsen, and Winskel in [15] where they have proposed abstract ways of capturing the notion of behavioral equivalence through open maps based bisimilarity and its logical counterpart — path bisimilarity. As shown in [15,18,22,5], bisimilarity induced by open maps makes possible a uniform definition of the numerous suggested behavioral equivalences (e.g., trace and testing equivalences,

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bisimulation, barbed and weak bisimulations, strong history preserving bisimulation, etc.) across a wide range of models for concurrency (e.g., transition systems, event structures, Petri nets, higher dimensional automata, etc.).

Another way to provide categorical characterizations is to adopt the coalgebraic approach which has both a field of its own interest presenting a deep mathematical foundation and a growing field of applications and interactions with various other approaches such as reactive and interactive system theory, object-oriented and concurrent programming, formal system specification, modal logic, etc. During the last years, it is becoming increasingly clear that a great variety of state-based dynamical systems, like transition systems, automata, process calculi and class-based systems can be captured uniformly as coalgebras. There is also a coalgebraic notion of bisimulation, the research in this area has been initiated by Aczel and Mendler [1]. Since then several papers have emerged in the literature (see [14,16,17,20,23,24] among others). One of the basic strands of the research is concerned with a coalgebraic rendering of various behavioral equivalences in the linear time – branching time spectrum.

The contribution of the paper is to show how several categorical (open maps, path-bisimilarity and coalgebraic) approaches to an abstract characterization of bisimulation relate to each other and to hereditary history preserving bisimulation [2,7] in the setting of HDA. Such a relating makes it possible to develop a metatheory designed for unified definition and study of equivalences in true concurrency semantics.

The rest of the paper is organized as follows. The basic concepts of HDA and their category are introduced in Section 2. Section 3 briefly recalls the notion of hereditary history preserving bisimulation in the setting of HDA. In Section 4, it is shown that the abstract equivalence based on spans of open maps coincides with the hereditary history preserving bisimulation. In the next section, we demonstrate how the equivalences under consideration can be captured by another category-theoretic bisimulation — path-bisimulation. Section 6 is devoted to a coalgebraic formulation of the equivalences. Proofs are omitted because of space limitations and can be found in a forthcoming paper.

2 Higher Dimensional Automata

In this section, we present the model of higher dimensional automata (HDA) – a geometric model for true concurrency based on the ideas of the works by V. Pratt [19] and R. van Glabbeek [6]. HDA are generalizations of the usual models of automata, also known as process graphs, state transition diagrams or labelled transition systems. The basic idea of HDA is to use the higher dimensions to represent the concurrent execution of processes. In contrast to interleaving models, HDA are built as sets of 0-cubes (points) and 1-cubes (edges) but also as sets of 2-cubes (squares), 3-cubes (cubes) and more generally n-cubes (hypercubes). In this way, an n-cube represents concurrent executions of n actions, whereas the edges of this cube depict the mutually exclusive execution of these n actions. Notice that 2-cubes are nothing but a local commutation relation as in

Mazurkiewicz trace theory, independence relation as in asynchronous transition systems, as in trace automata and as in transition systems with independence. In HDA, as shown on the left side of Figure 1, for two actions a and b, we model their concurrent execution by the square x labelled by $\{a, b\}$ and delineated by the 1-cubes x_1, y_1 and x_2, y_2 (in some sense, x_2 and y_2 are copies of x_1 and y_1, respectively). On the other hand, a mutually exclusive execution of a and b is modelled by the HDA generated by the 1-cubes x_1, y_1 and x_2, y_2 as shown on the right side of Figure 1. Thus, in HDA non-determinism arises as holes but concurrency is modelled by hypercubes with the interior filled. It is natural to graphically represent n-cubes as n-dimensional objects whose boundaries are the $(n-1)$-cubes from which the n-cubes can start and to which they end up. The 2-cube x shown on the left side of Figure 1 can start from x_1 or y_1, and, similarly, x ends up to x_2 or y_2. Hence, the boundary of the square can be divided into two source boundary functions: d_1^0 with $d_1^0(x) = x_1$ and d_2^0 with $d_2^0(x) = y_1$, and two target boundary functions: d_1^1 with $d_1^1(x) = x_2$ and d_2^1 with $d_2^1(x) = y_2$. In addition, we fix a distinguished basepoint called the *initial point* and denoted as i_0.

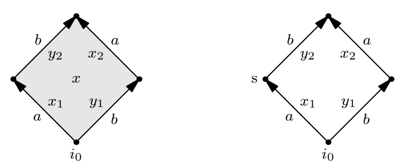

Figure 1. An example of concurrent and mutually exclusive executions of actions a and b in an HDA.

The following is the (well known but presented in a slightly different manner) definition of HDA from [7].

Definition 1. *A* precubical set *M is a collection of pairwise disjoint sets $(M_n)_{n\in\mathbb{N}}$ together with boundary functions $M_{n+1} \overset{d_i^0}{\underset{d_j^1}{\rightrightarrows}} M_n$ $(i, j = 1 \ldots (n+1))$ satisfying the commutativity of diagrams*

$$\begin{array}{ccc} M_{n+2} & \xrightarrow{d_j^m} & M_{n+1} \\ {\scriptstyle d_i^k}\big\downarrow & & \big\downarrow{\scriptstyle d_i^k} \\ M_{n+1} & \xrightarrow[d_{j-1}^m]{} & M_n \end{array}$$

for all $i < j$ and $k, m = 0, 1$.

Definition 2. *A* (labelled non-degenerate) HDA *(*over a set L of actions*) is a triple* $\mathrm{M} = (M, i_0^{\mathrm{M}}, l_L^{\mathrm{M}})$*, where*

- M *is a precubical set possessing the non-degeneracy property: for all* $x \in M_{n+1}$ *and* $m = 0, 1$ *it holds* $|\{d_i^m(x) \mid i = 1 \ldots n+1\}| = n + 1$,
- $i_0^{\mathrm{M}} \in M_0$ *is a distinguished basepoint of* M*, called the* initial point*,*
- $l_L^{\mathrm{M}} : M_1 \to L$ *is a* labelling function *from the 1-cubes of* M *to a set* L *of actions such that* $l_L^{\mathrm{M}}(d_i^0(x)) = l_L^{\mathrm{M}}(d_i^1(x))$ *for all* $i = 1, 2$ *and* $x \in M_2$.

If no confusion is possible we may ommit the subscripts and superscripts and write $\mathrm{M} = (M, i_0, l)$ to denote an HDA M over a set L of actions.

Remark 1. Assume $\mathrm{M} = (M, i_0, l)$ to be an HDA over L. For an n-cube $x \in M_n$ with $n > 1$ and $1 \le i \le n$, the 1-cubes $d_1^{m_1^i} \circ \ldots \circ d_{i-1}^{m_{i-1}^i} \circ d_{i+1}^{m_{i+1}^i} \circ \ldots \circ d_n^{m_n^i}(x)$, with $m_j^i = 0, 1$, $1 \le j \le n$ and $j \ne i$, represent the same action $l_i(x) = l(d_1^{m_1^i} \circ \ldots \circ d_{i-1}^{m_{i-1}^i} \circ d_{i+1}^{m_{i+1}^i} \circ \ldots \circ d_n^{m_n^i}(x))$, since $l(d_r^0(y)) = l(d_r^1(y))$ for all $y \in M_2$ and $r = 1, 2$. So, we can extend the labelling function to all cubes in M by taking for $x \in M_n$ an action $l(x) = (l_1(x), \ldots, l_n(x))$, if $n > 1$, and $l(x) = \emptyset$, if $n = 0$.

Example 1. To illustrate the concept specified in Definition 2, consider the HDA $\mathrm{M} = (M, i_0, l)$ over $L = \{a, b, c, d\}$, depicted in Figure 2. M contains the 3-cube x and the 2-cube y convoluted to the cylinder. To define the boundaries of x and y, we put $x_1 = d_1^1(x)$, $x_2 = d_2^0(x)$, $x_3 = d_3^1(x)$, $y_1 = d_1^0(y)$ and $y_2 = d_2^0(y)$. Clearly, M possesses the non-degeneracy property. The initial point is $i_0 \in M_0$. The actions of the edges of x and y are given by $l(d_2^0(d_3^0(x))) = a$, $l(d_1^0(d_3^0(x))) = b$, $l(d_1^0(d_2^0(x))) = c$ and $l(d_1^0(y)) = d$.

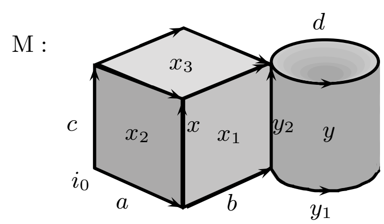

Figure 2. An example of an HDA M.

Define a morphism between two HDA mapping cubes and actions of the simulated system to simulating cubes and actions of the other and satisfying some requirements.

Definition 3. *Let* $\mathrm{M} = (M, i_0^{\mathrm{M}}, l_{L^{\mathrm{M}}}^{\mathrm{M}})$ *and* $\mathrm{N} = (N, i_0^{\mathrm{N}}, l_{L^{\mathrm{N}}}^{\mathrm{N}})$ *be HDA. A mapping* $\mathrm{f} = \langle f, \alpha \rangle$ *(where* $f = \cup f_n$*,* $f_n : M_n \to N_n$ *and* $\alpha : L^{\mathrm{M}} \to L^{\mathrm{N}}$*) is called a* morphism *from* M *to* N *iff it holds: 1.* $f_0(i_0^{\mathrm{M}}) = i_0^{\mathrm{N}}$*, 2.* $l_{L^{\mathrm{N}}}^{\mathrm{N}} \circ f = \alpha \circ l_{L^{\mathrm{M}}}^{\mathrm{M}}$*, 3.* $f_n \circ d_i^m = d_i^m \circ f_{n+1}$.

The first condition guarantees that morphisms preserve initial points; the second and third conditions ensure the consistency of actions and boundaries of cubes, respectively.

HDA with morphisms between them form a category **HDA** in which the composition of two morphisms $\mathrm{f} = \langle f, \alpha\rangle : \mathrm{M} \to \mathrm{M}'$ and $\mathrm{g} = \langle g, \beta\rangle : \mathrm{M}' \to \mathrm{M}''$ is $\mathrm{g} \circ \mathrm{f} = \langle g \circ f,\ \beta \circ \alpha\rangle : \mathrm{M} \to \mathrm{M}''$, and the identity morphism is a pair of the identity functions.

3 Hereditary history preserving bisimulation

In order to reason about the behaviour of HDA, we introduce the following notions and notations. A *cubical path* in an HDA M over L is a sequence $P = p_0 p_1 \dots p_k$[4] of cubes such that $p_{s-1} = d_i^0(p_s)$ or $p_s = d_j^1(p_{s-1})$ for all $s = 1 \dots k$, and, moreover, $p_0 = i_0^{\mathrm{M}}$. A cubical path $P = p_0 p_1 \dots p_k$ is *acyclic* if there are no other relations between the p_s and $p_{s'}$ $(0 \le s < s' \le k)$ than the relations above. For cubical paths $P = p_0 \dots p_k$ and $Q = q_0 \dots q_n$, we say that Q is an extension of P (denoted $P \to Q$) if $n \ge k$ and $p_0 \dots p_k = q_0 \dots q_k$. In particular, we write $P \xrightarrow{d_i^m} Q$ if $n = k+1$ and either $q_k = d_i^0(q_{k+1})$ for $m = 0$ or $q_{k+1} = d_i^1(q_k)$ for $m = 1$. Further, $\mathcal{CP}(\mathrm{M})$ ($\mathcal{CP}_p(\mathrm{M})$) is the set of all cubical paths (ending with a cube p) in M. For a cubical path $P = p_0 \dots p_k$ in an HDA $\mathrm{M} = (M, i_0, l)$ over L, define the structure $\mathrm{M}' = (M', i_0, l|_{(M')_1})$ with $(M')_n = \{d_{i_1}^{m_1} \circ \dots \circ d_{i_r}^{m_r}(p_s) \mid m_j = 0, 1\ (1 \le j \le r),\ 1 \le i_1 < \dots < i_r \le \dim p_s,\ 1 \le r \le \dim p_s,\ 1 \le s \le k\} \cup \{p_s \mid 0 \le s \le k\} \subseteq M_n$. It is easy to verify that M′ is an HDA over L, and, moreover, a sub-HDA of M. In this case, M′ is said to *have the form of the cubical path* P in the HDA M.

We proceed with some kind of equivalence on cubical paths [7]. A *homotopy* (denote $\sim$) is the least equivalence on cubical paths in M such that if P and P' are *s-adjacent* (denote $P \overset{s}{\leftrightarrow} P'$), i.e. P' can be obtained from P by replacing (for $i < j$ and $m = 0, 1$)

either a segment $\xrightarrow{d_i^0} p_s \xrightarrow{d_j^m}$ by a segment $\xrightarrow{d_{j-1}^m} p_s' \xrightarrow{d_i^0}$, or vice versa;

or a segment $\xrightarrow{d_j^m} p_s \xrightarrow{d_i^1}$ by a segment $\xrightarrow{d_i^1} p_s' \xrightarrow{d_{j-1}^m}$, or vice versa,

then P and P' are equivalent. Moreover, P and P' are (s, u, v)-*adjacent* (denote $P \overset{(s,u,v)}{\longleftrightarrow} P'$), if P' can be obtained from $P = \hat{p}_0 \dots \hat{p}_s \dots \hat{p}_k$ by an s-adjacency replacement of the segment $\xrightarrow{d_u^n} \hat{p}_s \xrightarrow{d_v^l}$. For every $P \in \mathcal{CP}(\mathrm{M})$ we write $[P]$ to denote its homotopy class.

[4] In case when a detailed presentation of P is needed, we shall write $P = p_0 \xrightarrow{d_{j_1}^{m_1}} \dots \xrightarrow{d_{j_k}^{m_k}} p_k$, where $d_{j_i}^{m_i}(p_i) = p_{i-1}$ if $m_i = 0$ and $d_{j_i}^{m_i}(p_{i-1}) = p_i$ if $m_i = 1$, for all $1 \le i \le k$.

Example 2. Consider the HDA M from Example 1. The sequences $P = i_0p_1p_2p_3x_1 y_2yp_7p_8p_7$ and $Q = i_0p_1p_2q_1q_2y_2yp_7p_8p_7$, shown in Figure 3, are cubical paths in M. The cubical paths are homotopic because $P \overset{4}{\leftrightarrow} (i_0p_1p_2q_1x_1y_2yp_7p_8p_7) \overset{5}{\leftrightarrow} Q$. An example of an acyclic cubical path is the sequence $i_0p_1p_2p_3x_1y_2$.

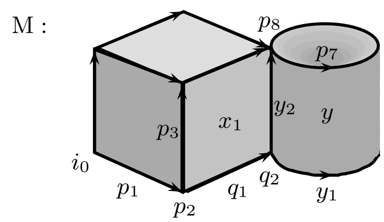

Figure 3. Cubical paths in the HDA M.

The following fact clarifies why the morphisms between HDA are simulations.

Lemma 1. *Given a morphism* $\mathrm{f} = \langle f, \alpha\rangle : \mathrm{M} \to \mathrm{N}$ *in* **HDA**, *for all* $P = p_0 \xrightarrow{d^{m_1}_{i_1}} \dots \xrightarrow{d^{m_k}_{i_k}} p_k \in \mathcal{CP}(\mathrm{M})$ *it holds:*

1. *there exists a unique* $f(P) = f(p_0) \xrightarrow{d^{m_1}_{i_1}} \dots \xrightarrow{d^{m_k}_{i_k}} f(p_k) \in \mathcal{CP}(\mathrm{N})$*;*
2. *whenever* $P \xrightarrow{d^m_i} P'$ *in* M, *then* $f(P) \xrightarrow{d^m_i} f(P')$ *in* N*;*
3. *whenever* $P \overset{(s,u,v)}{\longleftrightarrow} P'$ *in* M, *then* $f(P) \overset{(s,u,v)}{\longleftrightarrow} f(P')$ *in* N.

Further, we define a behavioural equivalence on HDA, called hereditary history preserving bisimulation (hhp-bisimulation), which is in close similarity with the corresponding definition from [7].

Definition 4. *Let* M *and* N *be HDA over* L.

Cubical paths $P = p_0 \dots p_k$ *in* M *and* $Q = q_0 \dots q_k$ *in* N *are called* l-related *iff* $l^{\mathrm{M}}(p_j) = l^{\mathrm{N}}(q_j)$ *for all* $j = 0 \dots k$.

A binary relation $\mathcal{R}$ *on cubical paths in* M *and* N *is called a* hereditary history preserving bisimulation (hhp-bisimulation) *between* M *and* N *if for any* $(P, Q) \in \mathcal{R}$, P *and* Q *are* l*-related and the following conditions are satisfied:*

1. *if* $P \xrightarrow{d^m_i} P'$ *then* $Q \xrightarrow{d^m_i} Q'$ *and* $(P', Q') \in \mathcal{R}$ *for some* Q' *in* N,
2. *if* $Q \xrightarrow{d^m_i} Q'$ *then* $P \xrightarrow{d^m_i} P'$ *and* $(P', Q') \in \mathcal{R}$ *for some* P' *in* M,
3. *if* $P' \to P$ *then* $Q' \to Q$ *and* $(P', Q') \in \mathcal{R}$ *for some* Q' *in* N,
4. *if* $Q' \to Q$ *then* $P' \to P$ *and* $(P', Q') \in \mathcal{R}$ *for some* P' *in* M,
5. *if* $P \overset{(s,u,v)}{\longleftrightarrow} P'$ *then* $Q \overset{(s,u,v)}{\longleftrightarrow} Q'$ *and* $(P', Q') \in \mathcal{R}$ *for some* Q' *in* N,
6. *if* $Q \overset{(s,u,v)}{\longleftrightarrow} Q'$ *then* $P \overset{(s,u,v)}{\longleftrightarrow} P'$ *and* $(P', Q') \in \mathcal{R}$ *for some* P' *in* M.

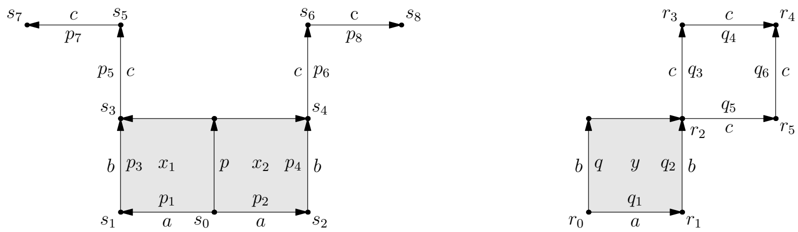

Figure 4. An example of hhp-bisimular HDA.

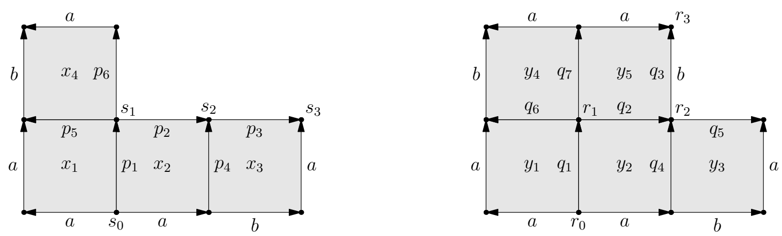

Figure 5. An example of non-hhp-bisimular HDA.

HDA M *and* N *are* hhp-bisimilar *if there exists an hhp-bisimulation between them which relates their initial points (regarded as cubical paths).*

Example 3. To get more intuition about the above concept, first contemplate the HDA shown in Figure 4. Let the boundary functions be given as follows: $d_1^0(x_1) = p_1$, $d_2^1(x_1) = p_3$, $d_1^0(x_2) = p_2$, $d_2^1(x_2) = p_4$ in the left-hand HDA, and $d_1^0(y) = q_1$, $d_2^1(y) = q_2$ in the right-hand HDA. Take the cubical paths $P_1 = s_0p_1s_1p_3s_3p_5s_5p_7s_7$ and $P_2 = s_0p_2s_2p_4s_4p_6s_6p_8s_8$ in the left-hand HDA and the cubical paths $Q_1 = r_0q_1r_1q_2r_2q_3r_3q_4r_4$ and $Q_2 = r_0q_1r_1q_2r_2q_5r_5q_6r_4$ in the right-hand HDA. It is easy to see that a required hhp-bisimulation $\mathcal{R}$ can be constructed from the set $\{(P_1, Q_1), (P_2, Q_2), (P_1, Q_2), (P_2, Q_1)\}$, using conditions 1-6 of Definition 4. Hence, these HDA are hhp-bisimilar. Next, examine the two HDA shown in Figure 5. Herein, the boundary functions are given as follows: $d_1^0(x_1) = p_1$, $d_2^1(x_1) = p_5$, $d_1^0(x_2) = p_1$, $d_2^1(x_2) = p_2$, $d_2^0(x_3) = p_4$, $d_1^1(x_3) = p_3$, $d_1^0(x_4) = p_5$, $d_2^0(x_4) = p_6$ in the left-hand HDA, and $d_1^0(y_1) = q_1$, $d_2^1(y_1) = q_6$, $d_1^0(y_2) = q_1$, $d_2^1(y_2) = q_2$, $d_2^0(y_3) = q_4$, $d_1^1(y_3) = q_5$, $d_1^0(y_4) = q_6$, $d_2^0(y_4) = q_7$, $d_2^0(y_5) = q_2$, $d_1^1(y_5) = q_3$ in the right-hand HDA. It is not difficult to see that the cubical path $(r_0q_1r_1q_2r_2q_3r_3)$ in the right-hand HDA can be related only to the cubical path $(s_0p_1s_1p_2s_2p_3s_3)$ in the left-hand HDA. Moreover, it holds that $(r_0q_1r_1q_2r_2q_3r_3) \overset{(5,1,1)}{\longleftrightarrow} (r_0q_1r_1q_2y_5q_3r_3)$ in the right-hand HDA. However, it is not the case that in the left-hand HDA there exists a cubical path P such that $(s_0p_1s_1p_2s_2p_3s_3) \overset{(5,1,1)}{\longleftrightarrow} P$. So, these HDA are not hhp-bisimilar.

4 Open Maps Bisimulation

In this section, we establish the coincidence between hhp-bisimulation and open maps based bisimulation in the context of HDA.

The concept of open map (open morphism) appears in work of Joyal and Moerdijk [13] where a notion of a subcategory of open maps of a (pre)topos is defined. As reported in [15,18], the open map approach provides general concepts of bisimilarity for any categorical model of computations. First, a category $\mathbb{M}$ whose objects represent models has to be identified. A morphism $f : X \longrightarrow Y$ in $\mathbb{M}$ should intuitively be thought of as a simulation of the object X in the object Y. Then, inside the category $\mathbb{M}$, a subcategory $\mathbb{P}$ of 'path objects' and 'path extension' morphisms between these objects is to be chosen. Given a path object P in $\mathbb{P}$ and a model object X in $\mathbb{M}$, a *path* is a morphism $p : P \longrightarrow X$ in $\mathbb{M}$. We think of p as representing a particular way of realizing P in X. Second, we have to identify morphisms $f : X \longrightarrow Y$ which have the property that whenever a computation of X can be extended via f in Y then that extension can be matched by an extension of the computation in X. A morphism $f : X \to Y$ in $\mathbb{M}$ is called $\mathbb{P}$-*open* iff whenever $m : P_1 \to P_2$ in $\mathbb{P}$, $p : P_1 \to X$ and $q : P_2 \to Y$ in $\mathbb{M}$, and $f \circ p = q \circ m$, then there exists a morphism $h : P_2 \to X$ in $\mathbb{M}$ such that $p = h \circ m$ and $q = f \circ h$. Third, an abstract notion of bisimilarity has to be introduced. The definition is given in terms of spans of open maps. Two objects X and Y in $\mathbb{M}$ are said to be $\mathbb{P}$-*bisimilar* if there exists a span $X \xleftarrow{f} Z \xrightarrow{f'} Y$ with a common object Z of $\mathbb{P}$-open morphisms.

We consider **HDA** as a category of models. For our purpose, we need to endow **HDA** with a fibred structure. Let $\mathbf{HDA}_L$ denote the subcategory of **HDA** whose objects are HDA labelled over L and morphisms have the identity action component. We shall follow similar conventions for the other categories defined in the paper. In the category **HDA**, we have to choose a full subcategory of path objects. For a natural number N, define the N-cube $\boxplus^N$ as $\{0\}$, if $N = 0$, and $\{(t_1, \ldots, t_N) \mid t_j \in \{0, \frac{1}{2}, 1\}\}$, otherwise. Clearly, the N-cube $\boxplus^N$ can be split into the sets $(\boxplus^N)_n = \{(t_1, \ldots, t_N) \in \boxplus^N \mid |\{t_j = \frac{1}{2} \mid 1 \le j \le N\}| = n\}$, where $0 \le n \le N$. For $(t_1, \ldots, t_N) \in \boxplus^N_n$, let the indexes $j_1 < \ldots < j_n$ be such that $t_{j_i} = \frac{1}{2}$ for all $1 \le i \le n$. Determine the boundary functions $\widetilde{d}_i^m$: $(\boxplus^N)_n \to (\boxplus^N)_{n-1}$ as follows: $\widetilde{d}_i^m(t_1, \ldots, t_N) = (t_1, \ldots, t_{j_i-1}, m, t_{j_i+1} \ldots, t_N)$, for all $m = 0, 1$, $1 \le i \le n$, and $0 < n \le N$. Obviously, $\boxplus^N$ is a precubical set possessing the non-degenerate property. Construct an HDA $\boxplus^{\mathrm{N}}$ over L as follows: $\boxplus^{\mathrm{N}} = (\boxplus^0, 0, \emptyset)$, if $N = 0$, and $\boxplus^{\mathrm{N}} = (\boxplus^N, (\underbrace{0, \ldots, 0}_N), l)$, otherwise (here, l is a labelling function from $(\boxplus^N)_1$ to a set L of actions, satisfying $l(\widetilde{d}_i^0(x)) = l(\widetilde{d}_i^1(x))$ for all $i = 1, 2$ and $x \in (\boxplus^N)_2$). A *path object* is an HDA having the form of a cubical path $\widetilde{P} \in \mathcal{CP}_p(\boxplus^{\mathrm{N}})$ such that $\widetilde{d}_1^1 \circ \cdots \circ \widetilde{d}_{\dim p}^1(p) = (\underbrace{1, \ldots, 1}_N)$, if $N > 0$.

We use **cP** to denote the full subcategory of the category **HDA**, whose objects are path objects. Clearly, the category $\mathbf{cP}_L$ is small, for a given set L of actions.

Our next aim is to characterize open morphisms in $\mathbf{HDA}_L$ relative to the subcategory $\mathbf{cP}_L$ defined prior to that. In the below characterization, the first condition is usually referred to as the "higher-dimensional" zig-zag property and the second one ensures that $\mathbf{cP}_L$-open morphisms reflect concurrency.

Theorem 1. *Given objects* M *and* M' *in* $\mathbf{HDA}_L$*, a morphism* $\mathrm{f} = \langle f, 1_L\rangle : \mathrm{M} \to \mathrm{M}'$ *is* $\mathbf{cP}_L$*-open iff for all* $P \in \mathcal{CP}(\mathrm{M})$ *the following holds:*

1. *if* $f(P) \xrightarrow{d_i^m} Q'$ *in* M'*, then* $P \xrightarrow{d_i^m} P'$ *and* $f(P') = Q'$ *for some* $P' \in \mathcal{CP}(\mathrm{M})$*,*
2. *if* $f(P) \overset{(s,u,v)}{\longleftrightarrow} Q'$ *in* M'*, then* $P \overset{(s,u,v)}{\longleftrightarrow} P'$ *and* $f(P') = Q'$ *for some* $P' \in \mathcal{CP}(\mathrm{M})$*.*

At last, the coincidence of $\mathbf{cP}_L$-bisimulation and hhp-bisimulation is established.

Theorem 2. *Two objects in* $\mathbf{HDA}_L$ *are* $\mathbf{cP}_L$*-bisimular iff they are hhp-bisimular.*

5 Path-Bisimulation

To obtain a logic characteristic of bisimulation induced by open maps, Joyal, Nielsen, and Winskel [15] have proposed a second category-theoretic characterization of bisimulation — path bisimulation which is a relation based generalization of open maps bisimulation.

Definition 5. *Let* $\mathbb{M}$ *be a category of models, let* $\mathbb{P}$ *be a small category of path objects, where* $\mathbb{P}$ *is a subcategory in* $\mathbb{M}$*, let* I *be a common initial object*[5] *in* $\mathbb{P}$ *and* $\mathbb{M}$*. Then,*

- *Two objects* X_1 *and* X_2 *in* $\mathbb{M}$ *are called* path-$\mathbb{P}$-bisimilar *iff there is a set* $\mathcal{B}$ *of pairs of paths* (p_1, p_2) *with common domain* P*, so* $p_1 : P \to X_1$ *is a path in* X_1 *and* $p_2 : P \to X_2$ *is a path in* X_2*, such that*
  - *(o)* $(i_1, i_2) \in \mathcal{B}$*, where* $i_1 : I \to X_1$ *and* $i_2 : I \to X_2$ *are the unique paths starting in the initial object, and for all* $(p_1, p_2) \in \mathcal{B}$ *and for all* $m : P \to Q$ *in* $\mathbb{P}$*, holds*
  - *(i) if there exists* $q_1 : Q \to X_1$ *with* $q_1 \circ m = p_1$ *then there exists* $q_2 : Q \to X_2$ *with* $q_2 \circ m = p_2$ *and* $(q_1, q_2) \in \mathcal{B}$ *and*
  - *(ii) if there exists* $q_2 : Q \to X_2$ *with* $q_2 \circ m = p_2$ *then there exists* $q_1 : Q \to X_1$ *with* $q_1 \circ m = p_1$ *and* $(q_1, q_2) \in \mathcal{B}$*.*
- *Two objects* X_1 *and* X_2 *in* $\mathbb{M}$ *are* strong path-$\mathbb{P}$-bisimilar *iff they are path-*$\mathbb{P}$*-bisimilar and the set* $\mathcal{B}$ *further satisfies:*
  - *(iii) If* $(q_1, q_2) \in \mathcal{B}$*, with* $q_1 : Q \to X_1$ *and* $q_2 : Q \to X_2$ *and* $m : P \to Q$ *in* $\mathbb{P}$*, then* $(q_1 \circ m, q_2 \circ m) \in \mathcal{B}$*.*

We are now ready to establish the main result of this section.

Theorem 3. $\mathbf{cP}_L$*-bisimulation, path-*$\mathbf{cP}_L$*-bisimulation, strong path-*$\mathbf{cP}_L$*-bisimulation all coincide with hhp-bisimulation.*

[5] In the case when $\mathbb{P}$ is $\mathbf{cP}_L$ and $\mathbb{M}$ is $\mathbf{HDA}_L$, the initial object is the HDA $\boxplus^0$.

6 Coalgebraic Bisimulation

Another alternative abstract characterization of bisimulation is based on a category of coalgebras induced by an endofunctor on an arbitrary category. In [16] it has been shown that the concept of path-bisimilarity can be translated into a coalgebraic setting with lax cohomomorphisms. Notice, in [20] a coalgebraic characterization of path-bisimilarity is obtained without the use of lax notions, however, in this case one cannot define a functor from a category of models of computations to the category of coalgebras.

We start with defining the terminology from [16]. Let $\mathbb{M}$ be a locally small category with a small path subcategory $\mathbb{P}$. We will define an embedding of $\mathbb{M}$ into a category of coalgebras for some endofunctor on the category $Set^{|\mathbb{P}|}$ of $|\mathbb{P}|$-sorted sets ($|\mathbb{P}|$-indexed sets), where $|\mathbb{P}|$ is the set of objects in $\mathbb{P}$. The endofunctor $F_{\mathbb{P}} : Set^{|\mathbb{P}|} \longrightarrow Set^{|\mathbb{P}|}$ is defined as follows:

$$\{X_P\}_{P\in|\mathbb{P}|} \longmapsto \{\prod_{Q\in|\mathbb{P}|} (\mathcal{P}(X_Q))^{Hom_{\mathbb{P}}(P,Q)}\}_{P\in|\mathbb{P}|},$$

where $\mathcal{P}(\cdot)$ denotes the powerset, X_P specifies a component of a $|\mathbb{P}|$-sorted set X for $P \in |\mathbb{P}|$, and $Hom_{\mathbb{P}}(P,Q)$ stands for the set of all morphisms from P to Q in $\mathbb{P}$. On morphisms in the category $Set^{|\mathbb{P}|}$, the endofunctor $F_{\mathbb{P}}$ acts by the following rule:

$$F_{\mathbb{P}} : (\{\gamma_P\}_{P\in|\mathbb{P}|} : X \to Y) \longmapsto \{\prod_{Q\in|\mathbb{P}|} h_Q^P\}_{P\in|\mathbb{P}|},$$

where $h_Q^P : \mathcal{P}(X_Q)^{Hom_{\mathbb{P}}(P,Q)} \to \mathcal{P}(Y_Q)^{Hom_{\mathbb{P}}(P,Q)} : g \longmapsto f,\ f(m) = \{\gamma_Q(x) \mid x \in g(m)\}$ for all $m \in Hom_{\mathbb{P}}(P,Q)$.

A *coalgebra for* $F_{\mathbb{P}}$ or $F_{\mathbb{P}}$-*coalgebra* is a pair (S, tr) with S an object in $Set^{|\mathbb{P}|}$ and $tr : S \to F_{\mathbb{P}}(S)$ a morphism in $Set^{|\mathbb{P}|}$, which consists of a family of functions:

$$\{tr_P : S_P \to \prod_{Q\in|\mathbb{P}|} (\mathcal{P}(S_Q))^{Hom_{\mathbb{P}}(P,Q)}\}_{P\in|\mathbb{P}|}.$$

The set S is called the *carrier* and the function tr is called the *coalgebra structure* of the $F_{\mathbb{P}}$-coalgebra.

A morphism $\gamma : S_1 \to S_2$ in the category $Set^{|\mathbb{P}|}$ is called a *cohomomorphism* between $F_{\mathbb{P}}$-coalgebras (S_1, tr_1) and (S_2, tr_2) iff $F_{\mathbb{P}}(\gamma) \circ tr_1 = tr_2 \circ \gamma$. $F_{\mathbb{P}}$-coalgebras and cohomomorphisms between them constitute a category, denoted by $\mathcal{CA}_{\mathbb{P}}$.

From now on, for an $F_{\mathbb{P}}$-*coalgebra* (S, tr), a triple $\langle m_1, m, m_2\rangle$, where $m_1 \in S_P$, $m_2 \in S_Q$ and $m \in Hom_{\mathbb{P}}(P,Q)$, satisfying $m_2 \in tr_P(m_1)(m)$, will be denoted by $m_1 \stackrel{m}{\to} m_2$.

As usual in the theory of coalgebras, bisimulation is a relation represented by a span of coalgebra morphisms [21]. An $F_{\mathbb{P}}$-*bisimulation* between two coalgebras (S_1, tr_1) and (S_2, tr_2) is a $|\mathbb{P}|$-sorted relation $R = \{R_P\}_{P\in|\mathbb{P}|} \subseteq (S_1 \times S_2)$ such that, if $(m_1, m_2) \in R_P$ and $m : P \to Q$ in $\mathbb{P}$, then

– if $m_1 \overset{m}{\to} m_1'$, then $m_2 \overset{m}{\to} m_2'$ and $(m_1', m_2') \in R_Q$ for some $m_2' \in S_2$,
– if $m_2 \overset{m}{\to} m_2'$, then $m_1 \overset{m}{\to} m_1'$ and $(m_1', m_2') \in R_Q$ for some $m_1' \in S_1$.

Next, following [16], we relax the requirement on coalgebra morphism. A morphism $\gamma : S \to S'$ in $Set^{|\mathbb{P}|}$ is called a *lax cohomomorphism* between $F_{\mathbb{P}}$-coalgebras (S, tr) and (S', tr') if for each $s \in S_P$ and $m \in Hom_{\mathbb{P}}(P, Q)$, $\{\gamma_Q(r) \mid r \in tr_P(s)(m)\} \subseteq tr'_P(\gamma_P(s))(m)$. $F_{\mathbb{P}}$-coalgebras and lax cohomomorphisms constitute a category, denoted by $\mathcal{CA}^{lax}_{\mathbb{P}}$ (the category $\mathcal{CA}_{\mathbb{P}}$ contains those lax cohomomorphisms for which the above inclusion is replaced by equality).

For $\mathbb{M}$ with $\mathbb{P}$, define a functor $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}} : \mathbb{M} \to \mathcal{CA}^{lax}_{\mathbb{P}}$. $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}$ acts on objects X in $\mathbb{M}$ as follows: $\{Hom_{\mathbb{M}}(P, X)\}_{P \in |\mathbb{P}|}$ is the carrier and $\{tr_P : m_1 \longmapsto \prod_{m \in \uplus_{Q \in |\mathbb{P}|} Hom_{\mathbb{P}}(P,Q)} \{m_2 \mid m_1 = m_2 \circ m\})\}_{P \in |\mathbb{P}|}$ is the coalgebra structure of the corresponding $F_{\mathbb{P}}$-coalgebra. $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}$ acts on morphisms $f : X \to Y$ in $\mathbb{M}$ as follows: $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(f)_P : Hom_{\mathbb{M}}(P, X) \to Hom_{\mathbb{M}}(P, Y) : \alpha \longmapsto (f \circ \alpha)$.

Proposition 1. *[16] For any two objects X and Y in $\mathbb{M}$, a $|\mathbb{P}|$-sorted relation R is a path-$\mathbb{P}$-bisimulation between X and Y iff it is an $F_{\mathbb{P}}$-bisimulation between $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(X)$ and $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(Y)$ containing the pair (i_X, i_Y), where $i_X : I \to X$ and $i_Y : I \to Y$ are paths, with an initial object I.*

Corollary 1. *For any two objects* M *and* M' *in* $\mathbf{HDA}_L$*,* $\mathbf{cP}_L$*-bisimulation, path-*$\mathbf{cP}_L$*-bisimulation, strong path-*$\mathbf{cP}_L$*-bisimulation, hhp-bisimulation coincide with* $F_{\mathbf{cP}_L}$*-bisimulation between* $\mathcal{B}eh^{\mathbf{HDA}_L}_{\mathbf{cP}_L}(\mathrm{M})$ *and* $\mathcal{B}eh^{\mathbf{HDA}_L}_{\mathbf{cP}_L}(\mathrm{M}')$ *containing the pair* $(i_{\mathrm{M}}, i_{\mathrm{M}'})$*, where* $i_{\mathrm{M}} : \boxplus^0 \to \mathrm{M}$ *and* $i_{\mathrm{M}'} : \boxplus^0 \to \mathrm{M}'$ *are paths, with the initial object* $\boxplus^0$*.*

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