

On the Finite Model Property of Infinitary Action Logics

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Abstract. We show that the equational theories of $*$ -continuous Kleene algebras, Kleene lattices, action algebras and action lattices have the finite model property (FMP). We present a uniform framework for proving this property for all these theories. We use the method of nuclei and quasi-embedding in the style of [14, 1], but we expand it to the infinitary Gentzen-style sequent calculi for these theories [3].

Keywords: Kleene algebra, Kleene lattice, action algebra, action lattice, $*$ -continuity, sequent calculus, finite model property, nucleus.

1 Introduction

One of the oldest and the most general logic of programs is the theory of Kleene algebras. The equational fragment of this theory was first studied by S. C. Kleene in the fifties of the last century under the name of the algebra of regular events or the algebra of regular sets [9]. Today we call the algebra studied by S. C. Kleene the algebra of regular languages, and we know that this algebra has no finite equational characteristic [18, 5]. Moreover we know the very smart quasi-axiomatization of Kleene algebras, and we know that the equational consequences of these axioms are exactly the identities of the algebra of regular languages (hence the regular expression equations). This is the famous result of D. Kozen from the early nineties [11] (we refer to his paper also for a brief history of those forty years of research on this topic).

By another result of D. Kozen [10], we know that exactly the same equations are satisfied by the algebra of binary relations. So Kleene algebras are good algebraic framework to reason about programs as well. Like in Propositional Dynamic Logic [6], 0 is interpreted as **abort**, 1 is interpreted as **skip**, $a \vee b$ means „nondeterministically run a or b ”, $a \cdot b$ means „sequentially run a and b ”, and a^* means „repeat a a nondeterministically chosen number of times”.

Let us recall the formal definition of Kleene algebras. An algebra $\mathbb{A} = (A, \vee, \cdot, *, 0, 1)$ is a *Kleene algebra* if the reduct $(A, \vee, 0)$ is a lower-bounded

join-semilattice, the reduct $(A, \cdot, 1)$ is a monoid, \cdot distributes over \vee on the both sides: $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$ and $(b \vee c) \cdot a = (b \cdot a) \vee (c \cdot a)$, 0 is two sided annihilator for \cdot : $a \cdot 0 = 0 = 0 \cdot a$, and the operation $*$ satisfies the following conditions:¹

$$\begin{aligned} 1 \vee a \cdot a^* &\leq a^*, & 1 \vee a^* \cdot a &\leq a^*, \\ a \cdot b \leq b &\Rightarrow a^* \cdot b \leq b, & b \cdot a \leq b &\Rightarrow b \cdot a^* \leq b. \end{aligned}$$

As usual a partial order relation \leq on A is defined by: $a \leq b \Leftrightarrow a \vee b = b$.

Shortly after the result of D. Kozen, V. Pratt showed that the regular expression equations are finitely equationally axiomatizable, but over an expanded signature [17]. He augmented the operators of Kleene algebras with two residuals for \cdot , i.e. binary operators \rightarrow and \leftarrow satisfying the residuation law:

$$b \leq a \rightarrow c \Leftrightarrow a \cdot b \leq c \Leftrightarrow a \leq c \leftarrow b.$$

We refer to this famous paper of V. Pratt for this equational characteristic. In this paper he also called the developed algebra an *action algebra* and the equational theory of this algebra *action logic*. So in action logic one can reason about programs in a purely equational way.

It is natural to supplement Kleene algebras and action algebras with the meet operator \wedge so that the reduct (A, \wedge, \vee) would be a lattice. We refer to the papers of D. Kozen [12] and P. Jipsen [8] for motivations of such extensions, mainly associated with the formation of matrices and the test operator. Similar to Kleene algebras and action algebras, Kleene lattices and actions lattices form respectively a quasi-variety and a variety.

We will denote the above-listed classes of algebras by: KA (Kleene algebras), KL (Kleene lattices), AA (action algebras) and AL (action lattices), and their equational theories respectively by Eq(KA), Eq(KL), Eq(AA) and Eq(AL).

The Kleene star operator $*$ in the standard Kleene algebras, i.e. the algebra of regular languages and the algebra of binary relations (which nota bene have naturally defined residuals and meet), has some essential infinitary property:

$$a \cdot b^* \cdot c = \sup_{n \in \omega} a \cdot b^n \cdot c,$$

where $b^0 = 1$ and $b^{n+1} = b \cdot b^n$. This property is called the **-continuity* condition and it can be captured by the following infinitary quasi-equation:

$$\bigwedge_{n \in \omega} a \cdot b^n \cdot c \leq d \Rightarrow a \cdot b^* \cdot c \leq d,$$

and the infinitely many equations: $a \cdot b^n \cdot c \leq a \cdot b^* \cdot c$, for each $n \in \omega$ [13].

We will denote the classes of **-continuous* algebras with the superscript $*$ and their equational theories in the similar manner, i.e.: by Eq(KA^{*}), Eq(KL^{*}), Eq(AA^{*}) and Eq(AL^{*}).

¹ We accept a convention assigning $*$ the highest priority, then \cdot , and at the end \vee .

2 State of the Art

By the FMP of the equational theory of some class of algebras we mean that if some equality fails in some member of this class it also fails in some finite member of this class. From this property it follows immediately that such theory is Π_1^0 , i.e. co-recursively enumerable. Clearly, when a theory has a finite axiomatization it is Σ_1^0 , i.e. recursively enumerable, and when it is Π_1^0 and Σ_1^0 it is decidable.

The table below shows what is currently known about the FMP and decidability of the equational theories from the previous section. Moreover it shows which theories coincide and which differ. Clearly, $\text{Eq}(C) \subseteq \text{Eq}(C^*)$ for all $C \in \{\text{KA}, \text{KL}, \text{AA}, \text{AL}\}$.

| theory | is decidable | has the FMP |
|---|--------------|-------------|
| $\text{Eq}(\text{KA})$ = $\text{Eq}(\text{KA}^*)$ | YES | YES |
| $\text{Eq}(\text{KL})$? $\text{Eq}(\text{KL}^*)$ | ? | ? |
| $\text{Eq}(\text{AA})$ ≠ $\text{Eq}(\text{AA}^*)$ | ? | NO |
| $\text{Eq}(\text{AL})$ ≠ $\text{Eq}(\text{AL}^*)$ | ? | NO |
| | NO | YES |

The identity of $\text{Eq}(\text{KA})$ and $\text{Eq}(\text{KA}^*)$ follows from the completeness theorem of D. Kozen [11]. Straightforwardly from this theorem it follows that these theories are PSPACE-complete, since the equality problem of regular expressions is PSPACE-complete. It is the famous result of L. J. Stockmeyer and A. R. Meyer from the early seventies [19]. The FMP of these theories is also a consequence of the completeness theorem of D. Kozen — one of the routine proof can be found in [15]. There is also much more significant result which states that the completeness theorem of D. Kozen is a consequence of the FMP of $\text{Eq}(\text{KA})$ in this paper of E. Palka [15].

As we know, none of the property from the table is known for $\text{Eq}(\text{KL})$ and $\text{Eq}(\text{KL}^*)$. We also do not know whether these theories coincide. In this paper we answer positively one question — does $\text{Eq}(\text{KL}^*)$ has the FMP?

All of the known answers concerning $\text{Eq}(\text{AA})$, $\text{Eq}(\text{AA}^*)$, $\text{Eq}(\text{AL})$ and $\text{Eq}(\text{AL}^*)$ are due W. Buszkowski [3]. W. Buszkowski reduced the total language problem for context-free grammars to $\text{Eq}(\text{AA}^*)$ and $\text{Eq}(\text{AL}^*)$ — it yields the Π_1^0 -hardness of these theories. Moreover, from another result of E. Palka [16] he deduced the

Π_1^0 -completeness and the FMP of these theories. From these facts it immediately follows that $\text{Eq}(\text{AA})$ and $\text{Eq}(\text{AL})$ (which clearly are Σ_1^0) are strictly contained in $\text{Eq}(\text{AA}^*)$ and $\text{Eq}(\text{AL}^*)$ respectively, and that they cannot have the FMP (since every finite algebra is $*$ -continuous). A summary of the results of W. Buszkowski and E. Palka and their extensions to variants of algebras of binary relations can be found in their joint paper [4]. Other results concerning these algebras and variants of algebras of regular languages can be found in the paper of W. Buszkowski [2].

3 Infinitary Action Logics

Most of the above-mentioned results of W. Buszkowski and E. Palka are based on the infinitary Gentzen-style system for $\text{Eq}(\text{AL}^*)$ developed by W. Buszkowski [3]. Our results are based on this system as well. Let us recall it. Atomic formulas of this system are variables and the constants 0 and 1. Formulas are formed out of atomic formulas by means of connectives: \wedge , \vee , \cdot , \rightarrow , \leftarrow and $*$. Sequents are expressions of the form $\Gamma \vdash A$, where Γ is a finite string of formulas and A is a formula. The axioms and inference rules are as follows:

$$(Id) \quad A \vdash A$$

$$(1R) \quad \vdash 1$$

$$(0L) \quad \Gamma \ 0 \ \Delta \vdash A$$

$$(1L) \quad \frac{\Gamma \ \Delta \vdash A}{\Gamma \ 1 \ \Delta \vdash A}$$

$$(\wedge L) \quad \frac{\Gamma \ A \ \Delta \vdash C}{\Gamma \ A \wedge B \ \Delta \vdash C} \qquad \frac{\Gamma \ B \ \Delta \vdash C}{\Gamma \ A \wedge B \ \Delta \vdash C}$$

$$(\wedge R) \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$(\vee L) \quad \frac{\Gamma \ A \ \Delta \vdash C \quad \Gamma \ B \ \Delta \vdash C}{\Gamma \ A \vee B \ \Delta \vdash C}$$

$$(\vee R) \quad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

$$(\cdot L) \quad \frac{\Gamma \ A \ B \ \Delta \vdash C}{\Gamma \ A \cdot B \ \Delta \vdash C}$$

$$(\cdot R) \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \ \Delta \vdash A \cdot B}$$

$$\begin{array}{l}
 (\rightarrow L) \quad \frac{\Gamma B \Delta \vdash C \quad \Psi \vdash A}{\Gamma \Psi A \rightarrow B \Delta \vdash C} \\
 (\rightarrow R) \quad \frac{A \Gamma \vdash B}{\Gamma \vdash A \rightarrow B} \\
 (\leftarrow L) \quad \frac{\Gamma B \Delta \vdash C \quad \Psi \vdash A}{\Gamma B \leftarrow A \Psi \Delta \vdash C} \\
 (\leftarrow R) \quad \frac{\Gamma A \vdash B}{\Gamma \vdash B \leftarrow A} \\
 (*L) \quad \frac{(\Gamma A^n \Delta \vdash B)_{n \in \omega}}{\Gamma A^* \Delta \vdash B} \\
 (*R) \quad \frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash A \quad \dots \quad \Gamma_n \vdash A}{\Gamma_1 \Gamma_2 \dots \Gamma_n \vdash A^*} \quad \text{for each } n \in \omega
 \end{array}$$

The $*$ -free fragment of this system is the variant of full Lambek calculus that is complete with respect to 0-bounded residuated lattices [7]. With the infinitary rule $(*L)$ and the infinitely many finitary rules $(*R)$ this system is complete with respect to $\text{Eq}(\mathbf{AL}^*)$ [16, 3]. Before we present our proof of completeness and simultaneously the FMP of this system we owe an explanation of the term A^n . Namely A^n stands for the string of n copies of A , and A^0 for the empty string ε . We will denote this system by \mathbf{AL}_ω^* (the subscript ω is used because of the infinitary rule $(*L)$).

Similarly to W. Buszkowski [3], we use the method of nuclei and quasi-embedding elaborated by M. Okada, K. Terui [14] and F. Belardinelli, P. Jipsen, H. Ono [1] for full Lambek calculi, but we expand it in a different way — without the usage of the $*$ -elimination theorem proved in [16] by a syntactic way. Therefore our proof is fully algebraic. Moreover our proof covers all interesting fragments of this system, i.e.: the \wedge -free fragment for actions algebras, the $\{\rightarrow, \leftarrow\}$ -free fragment for Kleene lattices and the $\{\wedge, \rightarrow, \leftarrow\}$ -free fragment for Kleene algebras. The listed fragments of this system we will denote respectively by \mathbf{AA}_ω^* , \mathbf{KL}_ω^* and \mathbf{KA}_ω^* . We expect that the method of W. Buszkowski can be applied to \mathbf{KA}_ω^* and \mathbf{KL}_ω^* as well, but it was done only for \mathbf{AL}_ω^* and \mathbf{AA}_ω^* .

We have to formally define an interpretation of sequents in an arbitrary $*$ -continuous Kleene/action algebra/lattice \mathbb{A} . Homomorphisms from the free algebra of formulas to \mathbb{A} are called assignments in \mathbb{A} . Every assignment f is extended to strings of formulas by setting:

$$f(\varepsilon) = 1, \quad f(A_1 A_2 \dots A_k) = f(A_1) \cdot f(A_2) \cdot \dots \cdot f(A_k).$$

A sequent $\Gamma \vdash A$ is said to be true in a model $\langle \mathbb{A}, f \rangle$ if $f(\Gamma) \leq f(A)$ in \mathbb{A} . This sequent is said to be true in \mathbb{A} if it is true in $\langle \mathbb{A}, f \rangle$ for any assignment f , and is

said to be valid (in the class \mathbf{C}^*) if it is true for every \mathbb{A} (from $\mathbf{C}^* \in \{\mathbf{KA}^*, \mathbf{KL}^*, \mathbf{AA}^*, \mathbf{AL}^*\}$). Indeed, according to this definition, all axioms are valid inequalities and all inference rules are valid quasi-inequalities of $*$ -continuous Kleene/action algebras/lattices.

We will present an example of a formal proof of one inequality (clearly, any formal proof of some equality $a = b$ consists of two formal proofs of the inequalities $a \leq b$ and $b \leq a$). Let us note that the sequent $\vdash A^*$ is an axiom — it is the rule $(*R)$ with 0 premises.

$$\frac{\frac{A \vdash A}{A \vdash A^*} (*R) \quad \frac{A \vdash A \quad A \vdash A}{A \vdash A^*} (*R) \quad \frac{A \vdash A \quad A \vdash A \quad A \vdash A}{A \vdash A^*} (*R) \quad \dots \quad}{(*)L} \quad \frac{\frac{A \ A^* \vdash A^*}{A \cdot A^* \vdash A^*} (\cdot L)}{\frac{1 \Rightarrow A^*}{1 \vee A \cdot A^* \Rightarrow A^*} (1L)}{\downarrow} (\vee L)$$

4 Finite Model Property

First of all we need to recall some notions and lemmas concerning the method of nuclei and quasi-embedding [14, 1, 7].

An operator $c: \wp(M) \rightarrow \wp(M)$ over a monoid $\mathbb{M} = (M, \cdot, 1)$ is called a *nucleus* if it satisfies the following conditions:

- (c1) $X \subseteq c(X)$,
- (c2) $X \subseteq Y \Rightarrow c(X) \subseteq c(Y)$,
- (c3) $c(c(X)) \subseteq c(X)$,
- (c4) $c(X) \circ c(Y) \subseteq c(X \circ Y)$,

for all $X, Y \subseteq M$, where $X \circ Y = \{x \cdot y \in M : x \in X, y \in Y\}$. A set $X \in \wp(M)$ is *closed* if $c(X) = X$. The set of all closed subsets of M is denoted by M_c .

Lemma 1. *Let $\mathbb{M} = (M, \cdot, 1)$ be a monoid and c be a nucleus over \mathbb{M} . Then the algebra*

$$\mathbb{L}_{c\mathbb{M}} = (M_c, \cap, \cup_c, \circ_c, \rightarrow, \leftarrow, c(\emptyset), c(\{1\}))$$

is a $c(\emptyset)$ -bounded residuated lattice, where $X \cup_c Y = c(X \cup Y)$, $X \circ_c Y = c(X \circ Y)$, $X \rightarrow Y = \{z \in M : X \circ \{z\} \subseteq Y\}$ and $X \leftarrow Y = \{z \in M : \{z\} \circ Y \subseteq X\}$. Moreover this lattice is complete, i.e.:

$$\inf \mathcal{X} = \begin{cases} \bigcap_{X \in \mathcal{X}} X & \text{if } \mathcal{X} \neq \emptyset, \\ M & \text{if } \mathcal{X} = \emptyset, \end{cases}$$

$$\sup \mathcal{X} = c(\bigcup_{X \in \mathcal{X}} X).$$

The residuated lattice \mathbb{L}_{cM} is called the *nuclear completion* of the monoid M . The partial order relation of \mathbb{L}_{cM} is \subseteq .

Another known fact from the field of residuated lattices is that the existence of residuals for \cdot implies that \cdot distributes over infinite joins. As a corollary we obtain that every complete residuated lattice can be expanded to a $*$ -continuous action lattice by setting:

$$a^* = \sup_{n \in \omega} a^n.$$

This corollary leads to the following extension of Lemma 1.

Lemma 2. *Let $M = (M, \cdot, 1)$ be a monoid and c be a nucleus over M . Then the algebra*

$$\mathbb{A}_{cM} = (M_c, \cap, \cup_c, \circ_c, \rightarrow, \leftarrow, *, c(\emptyset), c(\{1\}))$$

is an action lattice, where $X^ = c(\bigcup_{n \in \omega} X^n)$ ($X^0 = c(\{1\})$) and $X^{n+1} = X \circ_c X^n$ for $n \geq 0$).*

We will also need three more auxiliary lemmas which we formulate and prove here.

Lemma 3. *Let $M = (M, \cdot, 1)$ be a monoid and c be a nucleus over M . Then for all $X, Y \subseteq M$ there holds:*

$$X \circ_c Y = X \circ_c c(Y).$$

Proof. From (c1), (c2) and the monotonicity of \circ we yield that $X \circ_c Y \subseteq X \circ_c c(Y)$. On the other hand, in a similar way we obtain that $X \circ c(Y) \subseteq c(X) \circ c(Y)$ and by (c4) that $X \circ c(Y) \subseteq X \circ_c Y$. Finally, the thesis follows by (c2) and (c3).

Lemma 4. *Let $M = (M, \cdot, 1)$ be a monoid and c be a nucleus over M . Then for each $k > 1$ and for all $X_1, X_2, \dots, X_k \subseteq M$ there holds:*

$$X_1 \circ_c X_2 \circ_c \dots \circ_c X_k = c(X_1 \circ X_2 \circ \dots \circ X_k).$$

Proof. We will prove the lemma by induction on k . For the base step there is nothing to prove; therefore it suffices to show that

$$X_1 \circ_c X_2 \circ_c \dots \circ_c X_k = c(X_1 \circ X_2 \circ \dots \circ X_k)$$

for an arbitrary $k > 2$. From the induction hypothesis we have that

$$X_1 \circ_c X_2 \circ_c \dots \circ_c X_k = X_1 \circ_c c(X_2 \circ \dots \circ X_k),$$

where from the thesis follows by Lemma 3. □

Lemma 5. *Let $M = (M, \cdot, 1)$ be a monoid and c be a nucleus over M . Then for every family \mathcal{X} of subsets of M there holds:*

$$c(\bigcup_{X \in \mathcal{X}} c(X)) = c(\bigcup_{X \in \mathcal{X}} X).$$

Proof. By (c2), the inclusion $c(X) \subseteq c(\bigcup_{X \in \mathcal{X}} X)$ holds for every $X \in \mathcal{X}$, where from $c(\bigcup_{X \in \mathcal{X}} c(X)) \subseteq c(\bigcup_{X \in \mathcal{X}} X)$ follows. On the other hand, by (c1) and the monotonicity of \bigcup we have that $\bigcup_{X \in \mathcal{X}} X \subseteq \bigcup_{X \in \mathcal{X}} c(X)$. Finally, the thesis follows by (c2). \square

We can now present our extension of the method nuclei and quasi-embedding to the system \mathbf{AL}_ω^* and its fragments. We fix a sequent $\Gamma \vdash A$. By F we denote the set of all subformulas of formulas occurring in $\Gamma \vdash A$ plus the atomic formulas 0 and 1. By F^* we denote the set of all strings of formulas from F . Therefore, $\mathbb{F} = (F^*, \cdot, \varepsilon)$ is a monoid (\cdot stands for concatenation of strings).

In the method of nuclei and quasi-embedding we should now define the set \mathcal{T} as the set of all sequents which appear in the proof-search tree for $\Gamma \vdash A$. But such definition leads to an infinite set of sequents. So we need to define this set in a slightly different manner. Let \mathcal{T} be the smallest set of sequents satisfying the following conditions:

- $\Gamma \vdash A$ belongs to \mathcal{T} ,
- for any instance of the finitary inference rule of \mathbf{AL}_ω^* , if the conclusion of this rule belongs to \mathcal{T} then all premises of this rule belong to \mathcal{T} ,
- for any instance of the infinitary inference rule $(^*L)$, if the conclusion $\Psi C^* \Phi \vdash D$ belongs to \mathcal{T} then we choose to \mathcal{T} at most one premise $\Psi C^n \Phi \vdash D$ (for some $n \in \omega$) from the premises that are unprovable (i.e. if the conclusion is unprovable then we choose exactly one, and if it is provable we choose none).

We can treat this procedure as a reduction of the proof-search tree for $\Gamma \vdash A$ to a tree with nodes of finite degree. We will show now that all branches of this tree are finite as well, where from the finiteness of the set \mathcal{T} follows.

Let us recall some complexity measure of sequents from the paper of W. Buszkowski and E. Palka [4]. We will denote this measure by m . It assigns to every sequent the following sequence of integers:

$$m(\Delta \vdash B) = (u_1, u_2, \dots, u_r),$$

where r is the maximal complexity of formulas appearing in $\Delta \vdash B$, and u_i is the number of occurrences of formulas of complexity i in this sequent (the complexity of a formula is the total number of occurrences of symbols in it). We define a well-ordering relation \triangleleft on the set of all sequences of integers ω^* as follows:

$$\begin{aligned} (u_1, u_2, \dots, u_r) \triangleleft (v_1, v_2, \dots, v_s) \\ \Leftrightarrow \\ r < s \quad \text{or} \quad (r = s \quad \text{and} \quad u_{\max\{i:u_i \neq v_i\}} < v_{\max\{i:u_i \neq v_i\}}). \end{aligned}$$

It is easy to see that for any inference rule of \mathbf{AL}_ω^* the complexity of the conclusion is greater than the complexity of any premise. We can now prove the following lemma which is crucial for our extension of the method of nuclei and quasi-embedding.

Lemma 6. *All branches of the proof-search tree for $\Gamma \vdash A$ are finite.*

Proof. The proof is easy when it is proceeded by transfinite induction on $m(\Gamma \vdash A)$. The thesis straightforwardly follows from the construction of the proof-search tree for $\Gamma \vdash A$ and the induction hypothesis. \square

We can now define a relation $\preceq \subseteq F^* \times F$ as follows:

$$\Delta \preceq B \quad \Leftrightarrow \quad \Delta \vdash B \text{ is provable or } \Delta \vdash B \notin \mathcal{T}.$$

We should prove now that this relation is satisfied by all axioms and is preserved by all inference rules of \mathbf{AL}_ω^* . We will do it only for the rule $(*L)$, since for the remaining rules it follows from the corresponding proofs for full Lambek calculus, and for the rules $(*R)$ the proof is analogous.

Lemma 7. *If for all $n \in \omega$ holds $\Psi C^n \Phi \preceq D$, then $\Psi C^* \Phi \preceq D$ holds.*

Proof. Let $\Psi C^n \Phi \preceq D$ hold for all $n \in \omega$, and let us assume on the contrary that $\Psi C^* \Phi \preceq D$ does not hold. It means that the sequent $\Psi C^* \Phi \vdash D$ is not provable and it belongs to \mathcal{T} . Therefore, from the construction of the set \mathcal{T} , there exists one premise $\Psi C^n \Phi \vdash D$ in \mathcal{T} (for some $n \in \omega$) that is not provable. It means that $\Psi C^n \Phi \preceq D$ does not hold for this n . \square

The next step of the method of nuclei and quasi-embedding is a construction of the family of all basic sets \mathcal{B} as a family containing the following sets of strings of formulas for all $\Psi, \Phi \in F^*$ and $B \in F$:

$$[\Psi \Phi, B] = \{\Omega \in F^* : \Psi \Omega \Phi \preceq B\}.$$

We will denote any basic closed set of the form $[\varepsilon, B] = \{\Omega \in F^* : \Omega \preceq B\}$ by $[B]$ for short. Using this family we define an operator c over the monoid $\mathbb{F} = (F^*, \cdot, \varepsilon)$:

$$c(X) = \bigcap \{[\Psi \Phi, B] : X \subseteq [\Psi \Phi, B]\}.$$

This operator is nucleus over \mathbb{F} (it is the standard construction in the method of nuclei and quasi-embedding). As a corollary from this fact and Lemma 2 we obtain that the algebra

$$\mathbb{A}_{c\mathbb{F}} = (M_c, \cap, \cup_c, \circ_c, \rightarrow, \leftarrow, *, c(\emptyset), c(\{\varepsilon\}))$$

is an action lattice. Moreover we obtain that this algebra is finite since the family \mathcal{B} is finite (by the finiteness of the set \mathcal{T}).

The last technical lemma of the method of nuclei and quasi-embedding is a lemma on a quasi-embedding F in $\mathbb{A}_{c\mathbb{M}}$.

Lemma 8. *Let $f(p) = [p]$ be an assignment of variables from F in $\mathbb{A}_{c\mathbb{M}}$. Then:*

- for any formula $B \in F$, $B \in f(B) \subseteq [B]$,
- for any string of formulas $\Delta \in F^*$, $\Delta \in f(\Delta)$.

Proof. The proof proceeds by inductions on the complexity of B and Δ and it is analogous to the corresponding proof for full Lambek calculus. We present only cases concerning the constants 0 and 1 and the connectives \vee , \cdot and $*$.

- $B = 0$

First we have to show that $0 \in f(0) = c(\emptyset)$. According to the definition of c , $c(\emptyset)$ is the intersection of all sets from \mathcal{B} . But for any such set $[\Psi \Phi, G]$, $0 \in [\Psi \Phi, G]$, since $\Psi 0 \Phi \preceq G$ by the axiom (0L).

Second we have to show that $f(0) = c(\emptyset) \subseteq [0]$, however it is obvious by (c2) and (c3).

- $B = 1$

First we have to show that $1 \in f(1) = c(\{\varepsilon\})$. Let $[\Psi \Phi, G]$ be an arbitrary set such that $\{\varepsilon\} \subseteq [\Psi \Phi, G]$. Hence $\Psi \Phi \preceq G$, and since the relation \preceq is preserved by the rule (1L), $\Psi 1 \Phi \preceq G$ as desired.

Second we have to show that $f(1) = c(\{\varepsilon\}) \subseteq [1]$. Clearly, $\varepsilon \in [1]$, since the relation \preceq is satisfied by the axiom (1R). So, the thesis follows by (c2) and (c3).

- $B = C \vee D$

First we have to show that $C \vee D \in f(C \vee D) = f(C) \cup_c f(D)$. Let $[\Psi \Phi, G]$ be an arbitrary set such that $f(C) \cup f(D) \subseteq [\Psi \Phi, G]$, i.e. $f(C) \subseteq [\Psi \Phi, G]$ and $f(D) \subseteq [\Psi \Phi, G]$. From the induction hypotheses ($C \in f(C)$ and $D \in f(D)$) we yield respectively that $\Psi C \Phi \preceq G$ and $\Psi D \Phi \preceq G$. Finally we have that $\Psi C \vee D \Phi \preceq G$ by preserving the relation \preceq by the rule ($\vee L$).

Second we have to show that $f(C \vee D) = f(C) \cup_c f(D) \subseteq [C \vee D]$. It suffices to show that $f(C) \cup f(D) \subseteq [C \vee D]$, where from the thesis follows by (c2) and (c3). Let Ω be an arbitrary string of formulas such that $\Omega \in f(C) \cup f(D)$, i.e. $\Omega \in f(C)$ or $\Omega \in f(D)$. Let us assume that $\Omega \in f(C)$ (the second case is analogous), where from we yield that $\Omega \preceq C$ by the induction hypothesis $f(C) \subseteq [C]$. Finally we have that $\Omega \preceq C \vee D$ by preserving the relation \preceq by the rule ($\vee R$).

- $B = C \cdot D$

First we have to show that $C \cdot D \in f(C \cdot D) = f(C) \circ_c f(D)$. Let $[\Psi \Phi, G]$ be an arbitrary set such that $f(C) \circ f(D) \subseteq [\Psi \Phi, G]$, i.e. $E F \in [\Psi \Phi, G]$ for all $E \in f(C)$ and $F \in f(D)$. From the induction hypotheses ($C \in f(C)$ and $D \in f(D)$) we yield that $\Psi C D \Phi \preceq E$. Finally we have that $\Psi C \cdot D \Phi \preceq E$ by preserving the relation \preceq by the rule ($\cdot L$).

Second we have to show that $f(C \cdot D) = f(C) \circ_c f(D) \subseteq [C \cdot D]$. Analogously to the previous case it suffices to show that $f(C) \circ f(D) \subseteq [C \cdot D]$. Let Ω be an arbitrary string of formulas such that $\Omega \in f(C) \circ f(D)$, i.e. Ω is of the form $\Psi \Phi$ for some $\Psi \in f(C)$ and $\Phi \in f(D)$. From the induction hypotheses ($f(C) \subseteq [C]$ and $f(D) \subseteq [D]$) we yield that $\Psi \preceq C$ and $\Phi \preceq D$. Finally we have that $\Omega \preceq C \cdot D$ by preserving the relation \preceq by the rule ($\cdot R$).

- $B = C^*$

First we will show that

$$X^* = c(\bigcup_{n \in \omega} X^{\widehat{n}}), \quad (\sim)$$

where $X^{\widehat{0}} = \{\varepsilon\}$, $X^{\widehat{1}} = X$ and $X^{\widehat{n+1}} = X \circ X^{\widehat{n}}$ for $n \geq 1$.

Let us recall that

$$X^* = c(\bigcup_{n \in \omega} X^n),$$

where $X^0 = c(\{\varepsilon\})$ and $X^{n+1} = X \circ_c X^n$ for $n \geq 0$.

Therefore, $c(X^{\widehat{0}}) = X^0$, $c(X^{\widehat{1}}) = X^1$ (by Lemma 3) and $c(X^{\widehat{n}}) = X^n$ for each $n > 1$ (by Lemma 4). Hence, $X^* = c(\bigcup_{n \in \omega} c(X^{\widehat{n}}))$, where from (\wedge) follows by Lemma 5.

We will show now that $C^* \in f(C^*) = f(C)^*$. Let $[\Psi \Phi, G]$ be an arbitrary set such that $\bigcup_{n \in \omega} f(C)^{\widehat{n}} \subseteq [\Psi \Phi, G]$, i.e. for each $n \in \omega$, $f(C)^{\widehat{n}} \subseteq [\Psi \Phi, G]$. For $n = 0$ it means that $\{\varepsilon\} \subseteq [\Psi \Phi, G]$, i.e. $\Psi \Phi \preceq E$. For each $n > 0$, from the induction hypothesis ($C \in f(C)$) we yield that $\Psi C^n D \Phi \preceq E$. Finally we have that $\Psi C^* \Phi \preceq E$ by preserving the relation \preceq by the rule $(^*L)$.

Last we have to show that $f(C^*) = f(C)^* \subseteq [C^*]$. Analogously to the previous cases it suffices to show that $\bigcup_{n \in \omega} f(C)^{\widehat{n}} \subseteq [C^*]$. Let Ω be an arbitrary string of formulas such that $\Omega \in \bigcup_{n \in \omega} f(C)^{\widehat{n}}$, i.e. $\Omega \in f(C)^{\widehat{n}}$ for some $n \in \omega$. It means that Ω is of the form $\Psi_1 \Psi_2 \dots \Psi_n$ for some $\Psi_1, \Psi_2, \dots, \Psi_n \in f(C)$ when $n > 0$, or ε otherwise. From the induction hypothesis ($f(C) \subseteq [C]$) we yield that $\Psi_i \preceq C$ for each $i \in \{1, 2, \dots, n\}$. Finally we have that $\Omega \preceq C^*$ by preserving the relation \preceq by the rules $(^*R)$ (for $n = 0$, the axiom $(^*R)$). \square

Finally we can prove the main theorem of this paper, where from the completeness of the system \mathbf{AL}_ω^* with respect to $\mathbf{Eq}(\mathbf{AL}^*)$ and simultaneously the FMP of $\mathbf{Eq}(\mathbf{AL}^*)$ follow.

Theorem 1. *\mathbf{AL}_ω^* has the FMP.*

Proof. Assume that $\Gamma \vdash A$ is not provable. We define $\mathbb{A}_{c\mathbb{F}}$ and f as above. Then by Lemma 8, $\Gamma \in f(\Gamma)$. Since $\Gamma \vdash A$ belongs to \mathcal{T} , $\Gamma \preceq A$ does not hold. Hence, $\Gamma \notin [A]$, and again by Lemma 8, $\Gamma \notin f(A)$. So, $f(\Gamma)$ is not contained in $f(A)$, i.e., $\Gamma \vdash A$ is not true in the finite model $\langle \mathbb{A}_{c\mathbb{F}}, f \rangle$. \square

Clearly, this method can be applied to the subsystems \mathbf{AA}_ω^* , \mathbf{KL}_ω^* and \mathbf{KA}_ω^* , since the corresponding reducts of $\mathbb{A}_{c\mathbb{F}}$ are respectively a * -continuous action algebra, Kleene lattice and Kleene algebra. As a corollary we obtain that all the corresponding theories $\mathbf{Eq}(\mathbf{AA}^*)$, $\mathbf{Eq}(\mathbf{KL}^*)$ and $\mathbf{Eq}(\mathbf{KA}^*)$ have the FMP as well.

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