On Graded Nearness of Sets

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Abstract. In this article we present some mappings which can be used to measure the degree of inclusion of a set in a set and which characterize the binary relation of being-near-to, defined on finite sets of objects in line with Peters, Skowron, and Stepaniuk [12]. By means of these mappings, called inclusion functions in general, we extend the notion of nearness to the graded case where one can measure the degree to which one set is near to another one. We also give a number of basic properties of the notions introduced.

Keywords: nearness of sets, inclusion function, Pawlak rough approximation, granular computing.

1 Introduction

The starting point for our considerations is the concept of nearness of a set of objects to a set of objects in a Pawlak information system as proposed by J. F. Peters, A. Skowron, and J. Stepaniuk in their paper from 2007 [12]. Nearness of a set to a set has a topological flavour as a notion. It can be useful in granular computing, e.g. when investigating relationships between information granules like similarity or interaction.

In this paper we first characterize nearness of sets of objects by means of Pawlak upper rough approximation operators and the standard rough inclusion function. Then, some inclusion functions are defined to measure the degree of nearness of a set of objects to a set of objects. In other words, we extend in a few ways the notion of nearness, modelled by a binary relation on a family of sets of objects, to the concept of a graded nearness, modelled by a parameterized family of nearness-to-degree relations. We also investigate properties of the notions introduced. Proofs of the properties are rather straightforward and left as exercises.

The paper is organized as follows. In Sect. 2, the notion of an information granule and elements of rough approximation of sets in line with Pawlak are recalled. Also, the notion of nearness of sets is characterized by means of rough approximation operators. In Sect. 3, the notion of rough inclusion and its extensions are recalled. In Sect. 4, a few concepts of graded nearness of a set to a set of objects are introduced and properties of these concepts are presented. Section 5 summarizes the results.

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2 Nearness of Sets

In this section we recall Zadeh's concept of an information granule (infogranule for short), elements of rough approximation of sets in line with Pawlak, and the notion of nearness of sets as proposed by Peters, Skowron, and Stepaniuk in [12]. We also characterize the fact that a set is near to a set by means of rough approximation operators.

2.1 Similarity-based Information Granules

According to the definition proposed by Zadeh [20], an information granule (infogranule) is a clump of objects drawn together on the basis of indistinguishability, similarity or functionality. In this paper we will mainly consider infogranules being sets of objects of a given universe. Since the inner structure of infogranules is neglected, we may call them simple infogranules.

Given sets of objects U, U' and a relation $\varrho \subseteq U \times U'$, the image of a set $X \subseteq U$, $\varrho^{\rightarrow}(X)$, and the counter-image of a set $Y \subseteq U'$, $\varrho^{\leftarrow}(Y)$, are defined along the standard lines. In particular, for any $u \in U$ and $u' \in U'$,

$$\varrho^{\rightarrow}(\{u\}) = \{u' \in U' \mid (u, u') \in \varrho\} \ \& \ \varrho^{\leftarrow}(\{u'\}) = \{u \in U \mid (u, u') \in \varrho\}. \quad (1)$$

Now, let $U = U'$. Then, $\varrho^{\rightarrow}(X) = \varrho^{-1\leftarrow}(X)$ where ϱ^{-1} denotes the converse relation of ϱ. In our approach, similarity of elements of U is represented by a reflexive relation on U, so reflexive relations are also referred to as similarity relations.[3] Symmetric similarity relations are known as tolerance relations. Finally, indistinguishability of elements of U is represented by an equivalence relation on U. Therefore, equivalence relations are called indistinguishability (indiscernability) relations as well.

Let ϱ be a similarity relation on U and $u, u' \in U$. Then, $(u, u') \in \varrho$ is understood as u is similar to u'. The set $\varrho^{\leftarrow}(\{u\})$ (resp., $\varrho^{\rightarrow}(\{u\})$) consists of objects of U similar to u (to which u is similar). In this way, every object $u \in U$ is associated with a pair of infogranules $(\varrho^{\rightarrow}(\{u\}), \varrho^{\leftarrow}(\{u\}))$.[4] From our standpoint, such infogranules are particularly simple: we call them elementary infogranules.

2.2 A Pawlak-style Approximation of Sets

Here we briefly recall elements of rough approximation of sets in line with Pawlak. Given a universe of objects U and a similarity relation ϱ on it. Every subset of U may be approximated, e.g. by means of lower approximation operators

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[3] As argued by Tversky [17], modelling similarity of objects by a reflexive relation is a substantial simplification. A more realistic model should take into account not only arguments for something but also against.

[4] It is worth noting that $\varrho^{\rightarrow}(\{u\}) = \varrho^{\leftarrow}(\{u\})$ if ϱ is symmetric.

low, low* : $\wp U \mapsto \wp U$ and upper approximation operators upp, upp* : $\wp U \mapsto \wp U$, where $\wp U$ denotes the power set of U, such that for any $X \subseteq U$,

$$\begin{aligned} \mathrm{low}(X) &= \{u \in U \mid \varrho^{\leftarrow}(\{u\}) \subseteq X\}, \\ \mathrm{upp}(X) &= \{u \in U \mid \varrho^{\leftarrow}(\{u\}) \cap X \neq \emptyset\}, \\ \mathrm{low}^*(X) &= \{u \in U \mid \varrho^{\rightarrow}(\{u\}) \subseteq X\}, \\ \mathrm{upp}^*(X) &= \{u \in U \mid \varrho^{\rightarrow}(\{u\}) \cap X \neq \emptyset\}. \end{aligned} \tag{2}$$

The lower (resp., lower*) approximation of X, low(X) (low*(X)), may also be called the positive (positive*) region of X as usual. In the sequel, upp(X) and upp*(X) are referred to as the upper and upper* approximations of X and their set-theoretical complements are called the negative and negative* regions of X, respectively.

The above definition is a generalization of the definition of lower and upper rough approximation operators proposed by Pawlak, the inventor of rough sets [9, 10], to the case of similarity-based granulation of the universe of objects. Recall that indiscernability-based elementary infogranules were considered in the original approach of Pawlak. In such a case, due to symmetry of ϱ, low = low* and upp = upp*. In line with Pawlak we will say that X is exact (resp., exact*) if the boundary region of X, upp(X) − low(X) (resp., upp*(X) − low*(X)) is empty; otherwise, X is rough (rough*).

We first recall several properties of the approximation operators. For any $X \subseteq U$, X^c denotes the complement of X.

Proposition 1. *Let X, Y be any subsets of U and f, g denote* low, upp*, respectively or* low*, upp**, respectively.*

$$\begin{aligned}
&(a)\ \mathrm{upp}(X) = \varrho^{\rightarrow}(X)\ \ \&\ \ \mathrm{upp}^*(X) = \varrho^{\leftarrow}(X), \\
&(b)\ f(\emptyset) = g(\emptyset) = \emptyset\ \&\ f(U) = g(U) = U, \\
&(c)\ f(X) \subseteq X \subseteq g(X), \\
&(d)\ X \subseteq Y\ \Rightarrow\ f(X) \subseteq f(Y)\ \&\ g(X) \subseteq g(Y), \\
&(e)\ f(X \cap Y) = f(X) \cap f(Y)\ \&\ f(X) \cup f(Y) \subseteq f(X \cup Y), \\
&(f)\ g(X \cup Y) = g(X) \cup g(Y)\ \&\ g(X \cap Y) \subseteq g(X) \cap g(Y), \\
&(g)\ g(X)^c = f(X^c).
\end{aligned}$$

Example 1. Consider $U = \{u_0, \ldots, u_9\}$ and a similarity relation ϱ on U given by elementary infogranules of the forms $\varrho^{\leftarrow}(\{u\})$ and $\varrho^{\rightarrow}(\{u\})$ (see Table 1). Let $X_0 = \{u_3, u_4, u_5\}$, $X_1 = \{u_2, u_6, u_7\}$, and $X_2 = \{u_7\}$. Lower and upper approximations of these sets (and their *-versions) are given in Table 2.

2.3 Nearness of Sets vs. Rough Approximation

Given an information system (infosystem) $S = (U, A)$ where U is a non-empty finite set of objects and A is a non-empty finite set of attributes or features [8, 10, 11]. For simplicity assume that elements of A are mappings of the form

Table 1. Elementary infogranules induced by ϱ

u	$\varrho^{\leftarrow}(\{u\})$	$\varrho^{\rightarrow}(\{u\})$
u_0	$\{u_0, u_2\}$	$\{u_0, u_2, u_9\}$
u_1	$\{u_1, u_2\}$	$\{u_1, u_2\}$
u_2	$\{u_0, u_1, u_2, u_3\}$	$\{u_0, u_1, u_2, u_9\}$
u_3	$\{u_3, u_8, u_9\}$	$\{u_2, u_3, u_8\}$
u_4	$\{u_4, u_8\}$	$\{u_4, u_9\}$
u_5	$\{u_5, u_7\}$	$\{u_5, u_6\}$
u_6	$\{u_5, u_6, u_7\}$	$\{u_6, u_7\}$
u_7	$\{u_6, u_7, u_8\}$	$\{u_5, u_6, u_7, u_8\}$
u_8	$\{u_3, u_7, u_8, u_9\}$	$\{u_3, u_4, u_7, u_8\}$
u_9	$\{u_0, u_2, u_4, u_9\}$	$\{u_3, u_8, u_9\}$

Table 2. Examples of rough approximations

X_i	$\mathrm{low}(X_i)$	$\mathrm{low}^*(X_i)$	$\mathrm{upp}(X_i)$	$\mathrm{upp}^*(X_i)$
X_0	$\emptyset$	$\emptyset$	$\{u_2, \ldots, u_6, u_8, u_9\}$	$\{u_3, u_4, u_5, u_7, u_8, u_9\}$
X_1	$\emptyset$	$\{u_6\}$	$\{u_0, u_1, u_2, u_5, \ldots, u_9\}$	$\{u_0, \ldots, u_3, u_5, \ldots, u_8\}$
X_2	$\emptyset$	$\emptyset$	$\{u_5, \ldots, u_8\}$	$\{u_6, u_7, u_8\}$

$a : U \mapsto V_a$ where V_a is a set of values of a. The information provided by S about an object $u \in U$ takes the form of a signature of u understood as a set $\{(a, a(u)) \mid a \in A\}$.

Peters, Skowron, and Stepaniuk's idea of nearness described in [12] is that a set of objects X is near to a set of objects Y if there is an object of X whose signature is in some sense similar to the signature of an object of Y. Now suppose that ϱ reflects a considered kind of similarity of object signatures, i.e., for any objects $u, u' \in U$, $(u, u') \in \varrho$ if and only if the signature of u is similar in a given sense to the signature of u'. In this case, nearness of sets may be represented by a relation $\delta \subseteq \wp U \times \wp U$ such that for any $X, Y \subseteq U$,

$$(X, Y) \in \delta \overset{\mathrm{def}}{\Leftrightarrow} \exists u \in X.\exists u' \in Y.(u, u') \in \varrho \quad (3)$$

where $(X, Y) \in \delta$ reads as 'X is near to Y'. Let us note some basic properties of the nearness relation δ.

Proposition 2. *For any $X, Y, Z \subseteq U$, we have:*

(a) $(X,Y) \in \delta \;\Leftrightarrow\; \mathrm{upp}(X) \cap Y \neq \emptyset \;\Leftrightarrow\; X \cap \mathrm{upp}^*(Y) \neq \emptyset$,
(b) $(X,Y) \in \delta \;\Rightarrow\; \mathrm{upp}(X) \cap \mathrm{upp}(Y) \neq \emptyset \;\&\; \mathrm{upp}^*(X) \cap \mathrm{upp}^*(Y) \neq \emptyset$,
(c) $X \neq \emptyset \;\Rightarrow\; (X,X) \in \delta$,
(d) ϱ *is symmetrical* $\Rightarrow$ δ *is symmetrical* $\& \; ((X,Y) \in \delta \;\Leftrightarrow\; X \cap \mathrm{upp}(Y) \neq \emptyset)$,
(e) ϱ *is an equivalence relation* $\Rightarrow ((X,Y) \in \delta \;\Leftrightarrow\; \mathrm{upp}(X) \cap \mathrm{upp}(Y) \neq \emptyset)$,
(f) $(X,Y) \in \delta \;\&\; \mathrm{upp}^*(Y) \subseteq \mathrm{upp}^*(Z) \;\Rightarrow\; (X,Z) \in \delta$,
(g) $(X,Z) \in \delta \;\&\; \mathrm{upp}(X) \subseteq \mathrm{upp}(Y) \;\Rightarrow\; (Y,Z) \in \delta$,
(h) $(X \cup Y, Z) \in \delta \;\Leftrightarrow\; (X,Z) \in \delta$ *or* $(Y,Z) \in \delta$,
(i) $(X, Y \cup Z) \in \delta \;\Leftrightarrow\; (X,Y) \in \delta$ *or* $(X,Z) \in \delta$,
(j) $(X \cap Y, Z) \in \delta \;\Rightarrow\; (X,Z) \in \delta \;\&\; (Y,Z) \in \delta$,
(k) $(X, Y \cap Z) \in \delta \;\Rightarrow\; (X,Y) \in \delta \;\&\; (X,Z) \in \delta$.

Example 2. Given data from Example 1, one can see that X_0, X_1 are near to each other in the sense of δ. On the other hand, $(X_2, X_0) \in \delta$ but $(X_0, X_2) \notin \delta$.

3 Rough Inclusion and Its Extensions

Rough inclusion was introduced by Polkowski and Skowron as a fundamental notion of rough mereology (see, e.g., [13–15]).[5]

3.1 Rough Inclusion Functions (RIFs)

The idea of rough inclusion is realized, among other things, by rough inclusion functions (RIFs) which are mappings measuring the degree of inclusion of a set in a set. When viewed as ternary relations, they are generalizations of $\subseteq$ satisfying axioms of rough mereology. More precisely, a RIF over U is defined as a mapping $\kappa : \wp U \times \wp U \mapsto [0,1]$ such that rif_1 and rif_2^* below are true of κ:

$$\begin{aligned} &\mathrm{rif}_1(\kappa) \overset{\mathrm{def}}{\Leftrightarrow} \forall X, Y \subseteq U.(\kappa(X,Y) = 1 \;\Leftrightarrow\; X \subseteq Y), \\ &\mathrm{rif}_2^*(\kappa) \overset{\mathrm{def}}{\Leftrightarrow} \forall X, Y, Z \subseteq U.(\kappa(Y,Z) = 1 \;\Rightarrow\; \kappa(X,Y) \leq \kappa(X,Z)). \end{aligned} \tag{4}$$

Here is $\kappa(X,Y)$ understood as the degree of inclusion of X in Y. Observe that in the presence of $\mathrm{rif}_1(\kappa)$, the second postulate may be replaced by $\mathrm{rif}_2(\kappa)$ where

$$\mathrm{rif}_2(\kappa) \overset{\mathrm{def}}{\Leftrightarrow} \forall X, Y, Z \subseteq U.(Y \subseteq Z \;\Rightarrow\; \kappa(X,Y) \leq \kappa(X,Z)). \tag{5}$$

Examples of RIFs are the standard RIF, $\kappa^{£}$, which is the most popular among graded inclusion functions, κ_1 introduced by Gomolińska in [3], and κ_2 mentioned by Drwal and Mrózek in [2].[6]

[5] Rough mereology is a formal theory built upon Leśniewski's mereology [19].
[6] The notation used is ours. More details on properties and interrelationships among $\kappa^{£}$, κ_1, κ_2 can be found in [4].

3.2 The Standard RIF

Let U be finite. The cardinality[7] of a set X is denoted by $\#X$. The standard RIF over U, $\kappa^{£}$, is given by

$$\kappa^{£}(X,Y) \stackrel{\text{def}}{=} \begin{cases} \frac{\#(X\cap Y)}{\#X} & \text{if } X \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases} \tag{6}$$

In the theory of rough sets, the standard RIF is used from the 90s of the XX century but its inventor of unknown, at least up to the present authors' knowledge. The idea underlying $\kappa^{£}$ goes back to the notion of conditional probability and to Łukasiewicz's works on probability of truth of implicative formulas [7].

For any family of sets $\mathcal{X}$, we use $\mathrm{Pair}(\mathcal{X})$ to say that elements of $\mathcal{X}$ are pairwise disjoint. Let us recall basic properties of $\kappa^{£}$ (see, e.g., [4] for the proofs).

Proposition 3. *For any $X, Y, Z \subseteq U$ and a non-empty family $\mathcal{X}$ of subsets of U, we have:*

(a) $\kappa^{£}(X,Y) = 1 \Leftrightarrow X \subseteq Y$,

(b) $\kappa^{£}(Y,Z) = 1 \Rightarrow \kappa^{£}(X,Y) \leq \kappa^{£}(X,Z)$,

(c) $Z \subseteq Y \subseteq X \Rightarrow \kappa^{£}(X,Z) \leq \kappa^{£}(Y,Z)$,

(d) $\kappa^{£}(Y, \bigcup \mathcal{X}) \leq \sum_{X\in\mathcal{X}} \kappa^{£}(Y,X)$,

(e) $Y \neq \emptyset$ & $\mathrm{Pair}(\mathcal{X}) \Rightarrow \kappa^{£}(Y, \bigcup \mathcal{X}) = \sum_{X\in\mathcal{X}} \kappa^{£}(Y,X)$,

(f) $Y \neq \emptyset$ & $\mathrm{Pair}(\mathcal{X})$ & $\bigcup \mathcal{X} = U \Rightarrow \sum_{X\in\mathcal{X}} \kappa^{£}(Y,X) = 1$,

(g) $\kappa^{£}(\bigcup \mathcal{X}, Y) \leq \sum_{X\in\mathcal{X}} \kappa^{£}(X,Y) \cdot \kappa^{£}(\bigcup \mathcal{X}, X)$,

(h) $\mathrm{Pair}(\mathcal{X}) \Rightarrow \kappa^{£}(\bigcup \mathcal{X}, Y) = \sum_{X\in\mathcal{X}} \kappa^{£}(X,Y) \cdot \kappa^{£}(\bigcup \mathcal{X}, X)$,

(i) $X \neq \emptyset \Rightarrow (\kappa^{£}(X,Y) = 0 \Leftrightarrow X \cap Y = \emptyset)$,

(j) $X \neq \emptyset \Rightarrow \kappa^{£}(X,\emptyset) = 0$,

(k) $X \cap Y = \emptyset \Rightarrow \kappa^{£}(X, Z - Y) = \kappa^{£}(X, Z \cup Y) = \kappa^{£}(X,Z)$,

(l) $X \cap Y = \emptyset \Rightarrow \kappa^{£}(Z \cup Y, X) \leq \kappa^{£}(Z,X) \leq \kappa^{£}(Z - Y, X)$.

3.3 Remarks on Extensions of RIFs

It is easy to see that $\mathrm{rif}_1(\kappa)$ is equivalent to a conjunction of $\mathrm{rif}_0(\kappa)$ and $\mathrm{rif}_0^{-1}(\kappa)$ where

$$\mathrm{rif}_0(\kappa) \stackrel{\text{def}}{\Leftrightarrow} \forall X, Y \subseteq U.(X \subseteq Y \Rightarrow \kappa(X,Y) = 1),$$
$$\mathrm{rif}_0^{-1}(\kappa) \stackrel{\text{def}}{\Leftrightarrow} \forall X, Y \subseteq U.(\kappa(X,Y) = 1 \Rightarrow X \subseteq Y). \tag{7}$$

[7] In this case it is a number of elements.

Hence, we can obtain four classes of mappings extending RIFs. Namely, κ will be called a q-RIF over U ('q' for 'quasi') if $\mathrm{rif}_0(\kappa)$ and $\mathrm{rif}_2^*(\kappa)$ hold, whereas it will be called a weak q-RIF over U if it satisfies $\mathrm{rif}_0(\kappa)$ and $\mathrm{rif}_2(\kappa)$. Similarly, κ will be called a q'-RIF over U if $\mathrm{rif}_0^{-1}(\kappa)$ and $\mathrm{rif}_2^*(\kappa)$ hold true, and κ will be called a strong q'-RIF over U if $\mathrm{rif}_0^{-1}(\kappa)$ and $\mathrm{rif}_2(\kappa)$ are satisfied. One can easily prove that every RIF is both a q-RIF and a strong q'-RIF, every q-RIF is a weak q-RIF, and every strong q'-RIF is a q'-RIF. The inverse implications do not hold in general.

Examples of extensions of RIFs can be found, e.g. in [5, 6, 18]. In this article we only recall one q-RIF introduced in [5] and denoted here by $\kappa_{\mathrm{up}}^{\pounds}$ for simplicity. It is given for any subsets X, Y of a finite universe U by

$$\begin{aligned} \kappa_{\mathrm{up}}^{\pounds}(X,Y) &\overset{\mathrm{def}}{=} \kappa^{\pounds}(\mathrm{upp}(X), \mathrm{upp}(Y)), \text{ i.e.,} \\ \kappa_{\mathrm{up}}^{\pounds}(X,Y) &= \begin{cases} \frac{\#(\mathrm{upp}(X)\cap \mathrm{upp}(Y))}{\#\mathrm{upp}(X)} & \text{if } X \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{8}$$

3.4 Examples of Additional Postulates for Inclusion Functions

Apart from the basic postulates mentioned earlier, several other (optional) postulates for inclusion functions are considered in the rough set literature as, e.g., those presented below:

$$\begin{aligned} \mathrm{rif}_3(\kappa) &\overset{\mathrm{def}}{\Leftrightarrow} \forall X \subseteq U.(X \neq \emptyset \;\Rightarrow\; \kappa(X, \emptyset) = 0), \\ \mathrm{rif}_4(\kappa) &\overset{\mathrm{def}}{\Leftrightarrow} \forall X, Y \subseteq U.(\kappa(X,Y) = 0 \;\Rightarrow\; X \cap Y = \emptyset), \\ \mathrm{rif}_4^{-1}(\kappa) &\overset{\mathrm{def}}{\Leftrightarrow} \forall X, Y \subseteq U.(X \neq \emptyset \;\&\; X \cap Y = \emptyset \;\Rightarrow\; \kappa(X,Y) = 0), \\ \mathrm{rif}_6(\kappa) &\overset{\mathrm{def}}{\Leftrightarrow} \forall X, Y \subseteq U.(X \neq \emptyset \;\Rightarrow\; \kappa(X,Y) + \kappa(X, Y^c) = 1). \end{aligned} \tag{9}$$

In virtue of Proposition 3 $\kappa^{\pounds}$ satisfies all of them.[8]

4 A Graded Nearness of Sets

In this section we define two graded inclusion functions, κ' and κ''. By means of them and the mapping $\kappa_{\mathrm{up}}^{\pounds}$, mentioned earlier, we first characterize the relation of nearness δ. As we show next, each of the mappings induces a corresponding notion of graded nearness of sets to sets.

[8] Among RIFs it is an exception rather than a rule.

4.1 A Few Candidate Functions to Measure Nearness of Sets

Starting with a finite universe U and the standard RIF $\kappa^{£}$, let us define mappings $\kappa', \kappa'' : \wp U \times \wp U \mapsto [0,1]$ such that for any $X, Y \subseteq U$,

$$\begin{aligned}
\kappa'(X,Y) &\stackrel{\text{def}}{=} \kappa^{£}(\text{upp}(X), Y), \\
\kappa''(X,Y) &\stackrel{\text{def}}{=} \kappa^{£}(X, \text{upp}^*(Y)), \text{ i.e.,} \\
\kappa'(X,Y) &= \begin{cases} \frac{\#(\text{upp}(X) \cap Y)}{\#\text{upp}(X)} & \text{if } X \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases} \\
\kappa''(X,Y) &= \begin{cases} \frac{\#(X \cap \text{upp}^*(Y))}{\#X} & \text{if } X \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}
\end{aligned} \tag{10}$$

Now, we can characterize the nearness relation δ as follows.

Proposition 4. *For any $X, Y \subseteq U$ where $X \neq \emptyset$,*

(*a*) $(X,Y) \in \delta \Leftrightarrow \kappa'(X,Y) > 0 \Leftrightarrow \kappa''(X,Y) > 0$,
(*b*) *ϱ is an equivalence relation* $\Rightarrow ((X,Y) \in \delta \Leftrightarrow \kappa^{£}_{\text{up}}(X,Y) > 0)$.

Example 3. Given data from Example 1 we obtain $\kappa'(X_0, X_1) = 2/7$, $\kappa''(X_0, X_1) = 2/3$ (so, κ', κ'' yield different values in general), and $\kappa^{£}_{\text{up}}(X_0, X_1) = 5/7$. Moreover, $\kappa'(X_0, X_2) = \kappa''(X_0, X_2) = 0$ in accordance with the results from Example 2 and the above proposition. On the other hand, $\kappa^{£}_{\text{up}}(X_0, X_2) = 3/7 > 0$ which shows that $\kappa^{£}_{\text{up}}$ is not appropriate as a measure of nearness in the sense considered unless ϱ is an equivalence relation.

Several other properties of κ', κ'', and $\kappa^{£}_{\text{up}}$ are given below.

Proposition 5. *Let X, Y be any subsets of U and $\kappa \in \{\kappa', \kappa'', \kappa^{£}_{\text{up}}\}$. Then:*

(*a*) $\kappa'(X,Y) \leq \kappa^{£}_{\text{up}}(X,Y)$,
(*b*) $\text{rif}_0^{-1}(\kappa')$ & $\text{rif}_0(\kappa'')$ & $\text{rif}_0(\kappa^{£}_{\text{up}})$,
(*c*) $\text{rif}_2^*(\kappa')$ & $\text{rif}_2(\kappa'')$ & $\text{rif}_2^*(\kappa^{£}_{\text{up}})$,
(*d*) $\text{rif}_3(\kappa)$ & $\text{rif}_4(\kappa)$,
(*e*) $\text{rif}_6(\kappa')$.

Thus, in accordance with definitions from the previous section, κ' is a q'-RIF, κ'' is a weak q-RIF, and $\kappa^{£}_{\text{up}}$ is a q-RIF. It can be shown that rif_4^{-1} need not hold for the mappings considered. Similarly, rif_6 is not true of $\kappa'', \kappa^{£}_{\text{up}}$ in general.

4.2 Introducing Degrees of Nearness

Like in Ziarko's variable-precision rough set model [21, 22][9], where instead of one positive region of a set a family of variable-precision positive regions of the set

[9] See also more recent papers.

is considered, we can define the following parameterized families of relations of nearness-to-degree, viz., $\{\delta'_t\}_{t\in(0,1]}$, $\{\delta''_t\}_{t\in(0,1]}$, and $\{\delta_t\}_{t\in(0,1]}$ such that for any $X, Y \subseteq U$ and $t \in (0,1]$,

$$\begin{aligned}(X,Y) \in \delta'_t &\overset{\text{def}}{\Leftrightarrow} \kappa'(X,Y) \geq t,\\ (X,Y) \in \delta''_t &\overset{\text{def}}{\Leftrightarrow} \kappa''(X,Y) \geq t,\\ (X,Y) \in \delta_t &\overset{\text{def}}{\Leftrightarrow} \kappa(X,Y)^{\pounds}_{\text{up}} \geq t. \end{aligned} \tag{11}$$

Example 4. Continuing Example 3 let $t = 3/7$. It is easy to see that $(X_0, X_1) \in (\delta''_t \cap \delta_t) - \delta'_t$. More generally, $(X_0, X_1) \in \delta'_t$ for every $t \leq 2/7$ or, in words, X_0 is near to X_1 in the sense of δ'_t to degree t for all t not greater than $2/7$. Similarly, $(X_0, X_1) \in \delta''_t$ for every $t \leq 2/3$ and $(X_0, X_1) \in \delta_t$ for every $t \leq 5/7$.

Proposition 6. *For any* $X, Y, Z \subseteq U$, $\Delta_t \in \{\delta'_t, \delta''_t, \delta_t\}$, *and* $s, t \in (0,1]$, *we have:*

(a) $s \leq t \;\Rightarrow\; \Delta_t \subseteq \Delta_s$,
(b) $(\emptyset, Y) \in \Delta_t$,
(c) $X \neq \emptyset \;\&\; (X,Y) \in \delta'_t \cup \delta''_t \;\Rightarrow\; (X,Y) \in \delta$,
(d) $X \neq \emptyset \;\&\; \varrho$ *is an equivalence relation* $\&\; (X,Y) \in \delta_t \;\Rightarrow\; (X,Y) \in \delta$,
(e) $\kappa'(Y,Z) = 1 \;\&\; (X,Y) \in \delta'_t \;\Rightarrow\; (X,Z) \in \delta'_t$,
(f) $Y \subseteq Z \;\&\; \Delta_t \neq \delta'_t \;\&\; (X,Y) \in \Delta_t \;\Rightarrow\; (X,Z) \in \Delta_t$,
(g) $X \cap Y \neq \emptyset \;\Rightarrow\; \exists t \in (0,1].(X,Y) \in \Delta_t$,
(h) $X \neq \emptyset \;\Rightarrow\; \forall t \in (0,1].(X,\emptyset) \notin \Delta_t$,
(i) $\Delta_t \neq \delta'_t \;\&\; ((X,Y) \in \Delta_t$ *or* $(X,Z) \in \Delta_t) \;\Rightarrow\; (X, Y \cup Z) \in \Delta_t$,
(j) $\Delta_t \neq \delta'_t \;\&\; (X, Y \cap Z) \in \Delta_t \;\Rightarrow\; (X,Y) \in \Delta_t \;\&\; (X,Z) \in \Delta_t$,
(k) $\text{upp}(Y) \subseteq Y \cup Z \;\&\; (X,Y) \in \delta'_t \;\Rightarrow\; (X, Y \cup Z) \in \delta'_t$,
(l) $\text{upp}(Y \cap Z) \subseteq Y \;\&\; (X, Y \cap Z) \in \delta'_t \;\Rightarrow\; (X,Y) \in \delta'_t$.

Let us briefly comment on the properties. (a) says that the graded nearness notions are decreasing chains of relations. By (b) the empty set is near to any degree to any set of objects. (c) and (d) show the relationships between each of the graded nearness notions and the nearness in the sense of Peters, Skowron, and Stepaniuk. Properties (e) and (f) are relational counterparts of monotonicity in the second argument of graded inclusion functions. According to (g), if sets X and Y have a non-empty intersection, then X is to some degree near to Y. On the other hand, due to (h), non-empty sets cannot be near to any degree to the emptyset. The last four properties concern nearness to-degree of a set X to a set being a union or an intersection of some sets of objects.

5 Summary

Starting with the notion of nearness of a set to a set introduced by Peters, Skowron and Stepaniuk in 2007, we have introduced three concepts of graded nearness in the form of parameterized families of binary relations on the power set of a finite universe U. By help of these relations one can express the fact that a set of objects of U is to some degree near to a set of objects of U. In our proposal, graded nearness is defined by means of certain graded inclusion functions, whereas the latter are obtained from the standard rough inclusion and Pawlak upper approximation operators.

The concepts of nearness and graded nearness can be useful, e.g. in modelling of interaction in the context of granular computing and in construction of compound infogranules from simpler ones. Application of the notions proposed here to these problems will be our future research direction.

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